Money and Capital*

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Abstract

We analyze a microfounded model of monetary exchange with periodic centralized and decentralized markets, extended to include capital as a factor of production. Different from earlier attempts to integrate capital and monetary theory, our framework does not dichotomize: one cannot solve independently for equilibrium in the centralized and decentralized markets. Because of feedback across markets, money has interesting implications for investment, consumption, and employment. We calibrate and use the model to study quantitatively the effects of monetary and fiscal policy, taking into account long-run transitions. As an example, we find that the cost of 10% inflation can be between 1% and 4% of consumption, and that replacing inflation with taxes may be beneficial.

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1 Introduction

The goal of this paper is to extend, analyze, and study policy in, some models from the new monetary economics literature, both qualitatively and also quantitatively, as people have been routinely doing in, say, real business cycle theory for a long time (see Cooley 1995 for a representative sample). Our approach builds on the model with periodic meetings of centralized and decentralized markets in Lagos and Wright (2005), hereafter LW. However, for our purposes it is critical to extend that simple framework to include neoclassical firms employing labor and capital, as in the standard growth model. There are two main ways to motivate what we do and how we do it.

First, a previous attempt to integrate monetary and growth theory in Aruoba and Wright (2003) was at best partially successful, because that specification displays a strong *dichotomy*: one can solve independently for equilibrium allocations in the centralized and decentralized markets. This has some implications that seem undesirable, including the prediction that monetary policy can have no impact on investment, employment or consumption in the centralized market.[1] Although we discuss several ways to break the dichotomy, our baseline specification has some capital produced in the centralized market used as an input to decentralized market production. Then, since inflation is a tax on decentralized market activity, it influences the demand for capital, implying potentially rich feedback across markets. Now monetary policy can have interesting implications for investment, employment and consumption.

Second, there have been very few previous attempts to take microfounded monetary theories to the data, especially versions with capital.[2] It is because we are interested in quantitative analysis that we think it is important to include capital, which is an staple in mainstream macroeconomics. For the same reason, we include some other ingredients,

[1] Moreover, one might say that it implies money and growth theory are really not integrated at all. See Howitt (2003) and Waller (2003) for additional discussion.

[2] One exception is Shi (1999), who uses a very different approach. A few attempts to quantify simple search-based monetary models without capital are surveyed in Craig and Rocheteau (2005).

such as government spending and taxation. Incorporating these features is important for calibration purposes, and also allows us to analyze fiscal policy and combined monetary-fiscal policy experiments. In discussing policy, the analysis here is more challenging and certainly more interesting than in models without capital, because we need to take into account transition paths. Therefore we need to solve numerically for the equilibrium decision rules, as opposed to merely comparing steady states.

In terms of theory, our contribution is to construct a novel monetary model with capital used as a factor of production in both centralized and decentralized markets (as we discuss in detail below, capital is only a factor of production and not a medium of exchange). Again, this generates interesting feedback across markets and from money to all variables. We consider both the case where prices in the decentralized market are determined via bargaining, and the case where they are determined competitively. This allows us to highlight certain effects due to bargaining, referred to as *holdup problems*, on the demand for money and capital.[3] In addition to these effects, we have the usual monetary distortion from nominal interest rates, and distortions due to taxation. Even with all these effects, the model is tractable, and we show how to solve it explicitly in an example with common functional forms.

In terms of quantitative economics, we calibrate to mostly standard real observations, plus a few key monetary observations, including average velocity, the interest elasticity of money demand, and the inflation elasticity of investment. We do this for several specifications, including one that dichotomizes, and versions with bargaining as well as competitive pricing. Different specifications do a more or less reasonable job of capturing the key observations. We use the model to perform several policy experiments. For example, we measure the welfare gain of going from 10% to 0% inflation, with or without adjusting taxes to keep

[3]The holdup problem with money demand was discussed in LW. Holdup problems with investment are often thought to be an important factor influencing aggregate capital accumulation, but are difficult to incorporate into the standard growth model (Caballero and Hamour 1998; Caballero 1999). In our framework they rise naturally, due to the decentralized nature of some trades, and interact with policy in interesting ways. By comparing equilibrium with bargaining and with price taking, we can evaluate the importance of holdup problems qualitatively and quantitatively.

revenue constant, taking into account the long-run transition. Although naturally the answer depends on the version of the model and several other details, it is not difficult to find this gain can be as high 5% of consumption.

The paper is organized as follows. In Section 2 we describe our baseline model, and compare equilibria under bargaining and competitive pricing. In Section 3 we work through an explicit example, and analyze several extensions. In Section 4 we discuss calibration. In Section 5 we report the results of our policy experiments. In Section 6 we conclude.[4]

2 The Basic Model

There is a $[0,1]$ continuum of infinitely-lived agents. Time is discrete, and each period is divided into two subperiods. In one subperiod there is a frictionless or centralized market, referred to as the CM; in the other there is a decentralized market with two main frictions, referred to as the DM. The frictions are: (i) a double-coincidence problem, generated here by taste and technology shocks; and (ii) anonymity, which precludes private credit. This means that some medium of exchange is essential in the DM, as is standard in modern monetary theory (Kocherlakota 1998; Wallace 2001). The main issue in much of this literature (e.g. Kiyotaki and Wright 1989) is to determine endogenously which object will play this role. In order to focus on other questions, however, other papers avoid this issue by assuming there is a unique storable asset – perhaps fiat money, perhaps commodity money, perhaps something else – that qualifies as a potential medium of exchange.[5]

[4] A few words are perhaps in order as to why we use the LW framework. First, having some decentralized trade is what makes a medium of exchange essential. Then, having a periodic centralized market generates a huge gain in tractability over similar models without it, e.g. Green and Zhou (1998, 2002), Molico (1999), Zhou (1999), or Zhu (2003, 2005). This is because, with a centralized market, combined with quasi-linear preferences, we do not have to keep track of trading histories via the distribution of money holdings as a state variable. While models that do have to keep track of this distribution are clearly interesting, it is nice to have a benchmark that is easy to analyze and understand, the way e.g. the complete-market, representative-agent, neoclassical growth model serves as a benchmark in business cycle theory. Previous generations of search-based monetary theory, including Kiyotaki and Wright (1993), Shi (1995), or Trejos and Wright (1995), are also easy to analyze and understand, but mainly the assumptions seem to preclude quantitative work as it is normally practiced.

[5] For example, in Trejos and Wright (1995), it is assumed that "Agents consume *services* (or, equivalently, nonstorable goods)" to rule out commodity money, which is studied elsewhere, and concentrate on the role of fiat money in a model with search and bargaining. Of course, there is no constraint that agents have to

For the current project, we want to follow the latter approach and avoid the important but difficult problem of determining the medium of exchange endogenously. We cannot, however, assume there is a unique storable asset that qualifies for this role because we have money and capital. Our approach is to assume that capital is physically fixed in place in the CM, and thus cannot be traded in the DM. Then, to address the issue of why claims to (rather than physical units of) capital do not circulate in the DM, we assume that agents can costlessly counterfeit such claims, but cannot so easily counterfeit currency. Given this, sellers will never accept claims from anonymous buyers in the DM, but they may accept cash. So money is the only object that can serve as a medium of exchange in this environment, while capital is simply a productive input.[6]

We emphasize that we do not regard the approach outlined in the previous paragraph as a particularly interesting or elegant solution to the rate-of-return-dominance puzzle – how can money and other assets paying higher rates of return coexist? It is rather a device that allows us to study interactions between money and capital when one serves as medium of exchange and the other as a factor of production. Our position is that, even if we do not have a prize-winning answer to the rate-of-return-dominance question, for now, it is interesting to study other issues in models that include many of the ingredients from micro-based monetary economics, including double-coincidence problems, bargaining problems, and so on.

While we acknowledge that our assumptions about capital are crude, at the same time we insist that they are logically consistent assumptions about the physical environment, and not direct assumptions about agents' behavior. As a general principle, it should be clear that it is better to be explicit about the assumptions leading to an outcome, rather than assuming the outcome as a "reduced-form" for something left implicit. This is not

use fiat money in order to consume in that model – they are free to try direct barter if they like, and there generally is some barter in equilibrium.

[6] One need not interpret money here literally as cash. He, Huang and Wright (2005) show how to recast a closely related model so that agents can deposit their money in bank accounts in the CM and pay with checks or debit cards in the DM. We could do something similar, with little change except for an increase in notation. Hence, we will not introduce banks explicitly, but it is important to point out that one need not interpret money here as coins and currency, especially when it comes to quantitative analysis.

(or at least, not only) because some people may doubt that there exist logically consistent assumptions generating the outcome in question, but because one ought to want to know what *other* implications these assumptions may have. The only way to know this is to be explicit about the environment.[7]

To continue, in the CM there is a general good that can be use for consumption or investment. It is produced using labor H and capital K, hired by firms in perfectly competitive markets. As usual, profit maximization implies $r = F_K(K, H)$ and $w = F_H(K, H)$, where F is the technology, r is the rental rate and w the real wage, and by constant returns equilibrium profits will be 0. In the DM these firms do not operate, but an agent's own effort e and capital k may be used with technology $f(e, k)$. Note that k appears as an input in DM production, even though it cannot be produced or traded in the DM in our baseline model (it can be in one extension considered below). So while perhaps k cannot be physically moved to the location where the DM convenes, it still may increase productivity at that location (e.g. think about logging on to your computer from a remote site).

To generate a double-coincidence problem we adopt the following specification in the DM: with probability σ each agent wants to consume but cannot produce; with probability σ each agent can produce but does not want to consume; and with probability $1-2\sigma$ he can neither produce nor consume. This is equivalent for our purposes to the standard bilateral matching specification in the literature, where there is a probability σ of wanting to consume a good produced by a random partner. We frame things here in terms of taste and technology shocks, rather than matching, because it facilitates the discussion of alternative assumptions about DM pricing. Otherwise nothing hinges on this specification.

Instantaneous utility in the CM is $U(x) - Ah$, where x is consumption and h hours. In the

[7]It is of course interesting to try to understand the coexistence of currency and other assets at a deeper level. Lagos and Rocheteau (2005) discuss competition between money and capital as media of exchange in a related model. See also Waller (2003). There are other devices that could be used to capture why capital does not drive out money as a medium of exchange. One possibility is to introduce a small set of agents that do not accept capital in DM trades as a matter of exogenous policy, sometimes referred to as *government* agents, as in Aiyagari et al. (1994), Shi (2005), and Lagos (2005). Private information might also be useful, as in Williamson and Wright (1994), Trejos (1997), and Berentsen and Rocheteau (2004). We leave for future work further exploration of these ideas in other models with money and capital.

DM, with probability σ an agent is a consumer and his utility is $u(q)$, and with probability σ he is a producer and his utility is $-\ell(e)$, where q is consumption and e effort. Assume $U(x)$, $u(q)$ and $\ell(e)$ have the usual properties. Linearity in h is not important in principle, but it generates a huge gain in tractability, exactly as in LW.[8] Separability across (x, q, e) eases the presentation, but can be relaxed, and is in Section 3. It is convenient to write the disutility of effort in the DM as a disutility cost of production. Thus, given k, solve $q = f(e, k)$ for $e = \xi(q, k)$ and let $c(q, k) = \ell[\xi(q, k)]$. Notice $c_q > 0$, $c_k < 0$, $c_{qq} > 0$, and $c_{kk} > 0$ under the usual monotonicity and convexity assumptions on f and ℓ, and $c_{qk} < 0$ if $f_k f_{ee} < f_e f_{ek}$, which always holds if k is a normal input (see the Appendix).

There is a government that sets the money growth rate τ so that $M_{+1} = (1+\tau)M$, where a subscript $+1$ denotes next period; since the Fisher equation holds, this is equivalent to saying the government sets the nominal interest rate or the inflation rate. It also sets taxes on labor and capital income in the CM, t_h and t_k, as well as sales taxes in both the CM and DM markets, t_x and t_q. There is also a lump-sum tax T and government consumption G in the CM market. The government budget constraint is

$$G = T + t_h wH + t_k rK - \delta t_k K + t_x X + t_d \frac{\sigma M}{p} + \tau \frac{M}{p},$$

where δ is the depreciation rate on capital, which is tax deductible. Here p is the price level in the CM, so $t_d \sigma M/p$ is real DM sales-tax receipts, since σ is the number of trades and each trade will have M dollars changing hands in equilibrium. And $\tau M/p$ is seniorage revenue.

Agents discount between the CM and DM at rate β, but (to reduce notation) not between the DM and CM. If $W(m, k)$ and $V(m, k)$ are the value functions of agents in the CM and DM, then

$$\begin{aligned} W(m,k) &= \max_{x,h,m_{+1},k_{+1}} \{U(x) - Ah + \beta V(m_{+1}, k_{+1})\} \quad (1) \\ \text{s.t. } (1+t_x)\,x &= w\,(1-t_h)\,h + [1 + (r-\delta)\,(1-t_k)]\,k - k_{+1} - T + \frac{m - m_{+1}}{p} \end{aligned}$$

[8] We could alternatively assume general utility and indivisible labor, since as in Rogerson (1988) this gives rise to a quasi-linear reduced form; see Rocheteau et al. (2005) for the details.

Eliminating h using the budget equation, we have

$$\begin{aligned} W(m,k) &= \frac{A}{w\left(1-t_h\right)}\left\{\frac{m}{p}+\left[1+\left(r-\delta\right)\left(1-t_k\right)\right]k-T\right\} \\ &+\max_{x,m_{+1},k_{+1}}\left\{U(x)-\frac{A}{w\left(1-t_h\right)}\left[\frac{m_{+1}}{p}+\left(1+t_x\right)x+k_{+1}\right]+\beta V(m_{+1},k_{+1})\right\}. \end{aligned}$$

The FOC for an interior solution are:[9]

$$\begin{aligned} x &: U'(x)=\frac{A\left(1+t_x\right)}{w\left(1-t_h\right)} \\ m_{+1} &: \frac{A}{pw\left(1-t_h\right)}=\beta V_m(m_{+1},k_{+1}) \\ k_{+1} &: \frac{A}{w\left(1-t_h\right)}=\beta V_k(m_{+1},k_{+1}) \end{aligned} \tag{2}$$

Notice the choice of (m_{+1},k_{+1}) is independent of (m,k). Hence, given $V(m,k)$ is strictly concave, for any distribution of (m,k) across agents entering the CM, the distribution across agents leaving is degenerate (assuming an interior solution; see LW for assumptions to guarantee this is valid). We also have the envelope conditions:

$$W_m(m,k) = \frac{A}{pw\left(1-t_h\right)} \tag{3}$$

$$W_k(m,k) = \frac{A\left[1+\left(r-\delta\right)\left(1-t_k\right)\right]}{w\left(1-t_h\right)} \tag{4}$$

Hence, W is linear in (m,k).

Moving to the DM market, we have

$$V(m,k)=\sigma V^b(m,k)+\sigma V^s(m,k)+(1-2\sigma)W(m,k), \tag{5}$$

where

$$V^b(m,k) = u(q_b)+W\left(m-d_b,k\right) \tag{6}$$

$$V^s(m,k) = -c(q_s,k)+W\left[m+\left(1-t_q\right)d_s,k\right], \tag{7}$$

[9] The second order conditions are complicated because they involve second derivatives of V and hence third derivatives of u and c in models with bargaining. We simply assume V is strictly concave, but as in LW one can show this is true if the buyers' bargaining power is close to 1, or under additional conditions on preferences. In any case, for calibrated models one can check this numerically. In the model with price taking this is not an issue, since V is always concave.

while q_b and d_b are output and money exchanged when buying, and q_s and d_s when selling.[10] Using the linearity derived in (3), we have

$$V(m,k) = W(m,k) + \sigma\left[u(q_b) - \frac{d_b A}{pw(1-t_h)}\right] + \sigma\left[\frac{d_s A}{pw(1-t_h)} - c(q_s,k)\right]. \quad (8)$$

Differentiation yields

$$\begin{aligned} V_m(m,k) &= \frac{A}{pw(1-t_h)} + \sigma\left[u'\frac{\partial q_b}{\partial m} - \frac{A}{pw(1-t_h)}\frac{\partial d_b}{\partial m}\right] \\ &\quad + \sigma\left[\frac{A(1-t_q)}{pw(1-t_h)}\frac{\partial d_s}{\partial m} - c_q\frac{\partial q_s}{\partial m}\right] \quad (9) \\ V_k(m,k) &= \frac{A\left[1+(r-\delta)(1-t_k)\right]}{w(1-t_h)} + \sigma\left[u'\frac{\partial q_b}{\partial k} - \frac{A + A(r-\delta)(1-t_k)}{w(1-t_h)}\frac{\partial d_b}{\partial k}\right] \\ &\quad + \sigma\left[\frac{A + A(r-\delta)(1-t_k)}{w(1-t_h)}\frac{\partial d_s}{\partial k} - c_q\frac{\partial q_s}{\partial k} - c_k\right]. \quad (10) \end{aligned}$$

It remains to specify how the terms of trade (q,d) are determined, so that we can substitute for the derivatives in (9) and (10); this will differ across versions of the model considered below. Before pursuing equilibrium, however, consider the planner's problem in an economy *without* anonymity, so that money is *not* essential:

$$\begin{aligned} J(K) &= \max_{q,X,H,K_{+1}} \left\{U(X) - AH + \sigma\left[u(q) - c(q,K)\right] + \beta J(K_{+1})\right\} \quad (11) \\ s.t.\ X &= F(K,H) + (1-\delta)K - K_{+1} - G \end{aligned}$$

Eliminating X, the FOC are:

$$\begin{array}{ll} q: & u'(q) = c_q(q,K) \\ H: & A = U'(X)F_H(K,H) \\ K_{+1}: & U'(X) = \beta J'(K_{+1}) \end{array} \quad (12)$$

Using $J'(K) = U'(X)[F_K(K,H) + 1 - \delta] - \sigma c_k(q,K)$, we have

$$U'(X) = \beta U'(X_{+1})[F_K(K_{+1},H_{+1}) + 1 - \delta] - \beta\sigma c_k(q_{+1},K_{+1}). \quad (13)$$

From the first condition in (12), given K, we have $q = q^*(K)$ where $q^*(K)$ solves $u'(q) = c_q(q,K)$. Given this, the paths for (K_{+1},H,X) satisfy the the Euler equation (13), the

[10] Notice the DM sales tax t_q shows up in (7). One might wonder how government can tax transactions in an anonymous market. One answer is that for money to be essential we really only need buyers to be anonymous, and we can allow government to monitor sellers' receipts with no problem. More pragmatically, eliminating t_q does not affect the quantitative results much, and so we left it in.

second equation in (12), and the constraint in (11). These are all fairly standard, except for the presence of the term $-\beta\sigma c_k(q_{+1}, K_{+1}) > 0$ in (13), which reflects the fact that in general investment not only affects CM productivity but also DM productivity. If K did not appear in the DM technology, this term would vanish and the planner's solution would dichotomize: he can first set $q = q^*$, where q^* is defined by $u'(q) = c'(q)$, and then solve the other conditions independently for (K_{+1}, H, X).

2.1 Bargaining

Suppose each agent with a desire to consume in the DM is paired with one who can produce. Since buyers are anonymous, trade must be quid pro quo, and here this means they must pay with money. Let the buyer's and seller's states be (m_b, k_b) and (m_s, k_s). Then the terms of trade (q, d) solve the generalized Nash solution, with bargaining power for the buyer given by θ and threat points given by continuation values. The buyer's payoff from the trade is $u(q) + W(m_b - d, k_b)$ and his threat point $W(m_b, k_b)$, so (3) implies his surplus is $u(q) - \frac{A}{pw(1-t_h)}d$. Similarly, the seller's surplus is $\frac{A(1-t_q)}{pw(1-t_h)}d - c(q, k_s)$. Hence our bargaining solution is

$$\max_{q,d} \left[u(q) - \frac{Ad}{pw\,(1-t_h)}\right]^{\theta} \left[\frac{A\,(1-t_q)\,d}{pw\,(1-t_h)} - c(q, k_s)\right]^{1-\theta} \quad \text{s.t. } d \le m_b.$$

Exactly as in LW, one can show that in any equilibrium $d = m_b$. This implies $q \le q^*(k_s)$ where $q^*(k_s)$ is the solution to $u'(q) = c_q(q, k_s)$, and typically the inequality is strict.[11] In any case, inserting $d = m_b$ and taking the FOC wrt q, we get

$$\frac{m_b}{p} = \frac{g(q, k_s)w\,(1-t_h)}{A}, \tag{14}$$

where

$$g(q, k_s) \equiv \frac{\theta c(q, k_s)u'(q) + (1-\theta)u(q)c_q(q, k_s)}{(1-t_q)\,\theta u'(q) + (1-\theta)c_q(q, k_s)}. \tag{15}$$

We write $q = q(m_b, k_s)$, where $q(\cdot)$ is given by solving (14) for q as a function of (m_b, k_s) (the dependence of q on prices, taxes, and θ is implicit). Now one can compute $\partial d/\partial m_b = 1$,

[11] In models without capital, e.g., $q < q^*$ unless $\theta = 1$ and the nominal interest rate is 0.

$\partial q/\partial m_b = A/pw\,(1-t_h)\,g_q > 0$ and $\partial q/\partial k_s = -g_k/g_q > 0$, where

$$g_q = \frac{c_q u'[(1-t_q)\,\theta u' + (1-\theta)c_q] + \theta(1-\theta)[(1-t_q)\,u - c][u'c_{qq} - c_q u'']}{[(1-t_q)\,\theta u' + (1-\theta)c_q]^2} > 0 \quad (16)$$

$$g_k = \frac{\theta c_k u'\,[(1-t_q)\,\theta u' + (1-\theta)c_q] + \theta c_{qk}(1-\theta)u'\,[(1-t_q)\,u - c]}{[(1-t_q)\,\theta u' + (1-\theta)c_q]^2} < 0, \quad (17)$$

while the other derivatives in (9) and (10) are 0.

Inserting these derivatives and imposing $(m,k) = (M,K)$, (9) and (10) reduce to

$$V_m(M,K) = \frac{(1-\sigma)A}{pw\,(1-t_h)} + \frac{\sigma A u'(q)}{pw\,(1-t_h)\,g_q(q,K)} \quad (18)$$

$$V_k(M,K) = \frac{A + A\,(r-\delta)\,(1-t_k)}{w\,(1-t_h)} - \sigma\gamma(q,K), \quad (19)$$

where it is understood that $q = q(M,K)$, and[12]

$$\gamma(q,K) \equiv \frac{c_k(q,K)g_q(q,K) - c_q\,(q,K)\,g_k(q,K)}{g_q(q,K)} < 0.$$

Substituting (18) and (19), as well as prices $p = AM/w\,(1-t_h)\,g(q,K)$, $r = F_K(K,H)$, and $w = F_H(K,H)$, into the FOC for m_{+1} and k_{+1} in (2), we get the equilibrium conditions

$$\frac{g(q,K)}{M} = \frac{\beta g(q_{+1},K_{+1})}{M_{+1}}\left[1 - \sigma + \sigma\frac{u'(q_{+1})}{g_q(q_{+1},K_{+1})}\right] \quad (20)$$

$$U'(X) = \beta U'(X_{+1})\left\{1 + [F_K(K_{+1},H_{+1}) - \delta]\,(1-t_k)\right\} \quad (21)$$

$$-\sigma\beta\,(1+t_x)\,\gamma(q_{+1},K_{+1}).$$

The other two equilibrium conditions come from the FOC for X and the resource constraint,

$$U'(X) = \frac{A\,(1+t_x)}{(1-t_h)\,F_H(K,H)} \quad (22)$$

$$X + G = F(K,H) + (1-\delta)K - K_{+1}. \quad (23)$$

An *equilibrium* can now be defined as (positive, bounded) paths for (q, K_{+1}, H, X) satisfying (20)-(23), given policy and initial K_0.[13] When $M_{+1} = (1+\tau)\,M$ for fixed τ, a *steady*

[12] The term $-\sigma\gamma(q,K)$ in (19) is the marginal value of capital in the DM, which in general depends on the bargaining solution, and in particular on θ. Note that $\gamma = c_k + c_q\partial q/\partial K$, where the first term is the cost saving from having more capital and the second is the cost increase from having to produce more when you have more capital. It is this second term that captures the capital holdup problem.

[13] We focus on monetary equilibria, where $q > 0$. A nonmonetary equilibrium satisfies $q = 0$ instead of (20), (21) with $\gamma = 0$, and (22)-(23). Note that these are *exactly* the equilibrium conditions for the standard nonmonetary model in Hansen (1985). Also note that one can always give a more comprehensive definition of equilibrium, as in LW, including more general descriptions of decision rules, payoffs, and distributions; there is little to be gained from such pedantics here.

state is a constant solution (q, K, H, X) to (20)-(23). This means inflation equals τ, and if we define ρ by $\beta = \frac{1}{1+\rho}$ and the nominal interest rate by the Fisher equation $i = (1+\rho)(1+\tau)-1$, in steady state (20)-(21) simplify to

$$\frac{i}{\sigma} = \frac{u'(q)}{g_q(q,K)} - 1 \tag{24}$$

$$\rho = [F_K(K,H) - \delta](1-t_k) - \sigma(1+t_x)\frac{\gamma(q,K)}{U'(x)}. \tag{25}$$

Note for future reference that the price level in the DM is defined implicitly by $\tilde{p} = M/q$, which is to be contrasted with the price level p in the CM.

A very special case of this model is the specification in Aruoba and Wright (2003), where capital is not used in the DM, so $c(q,K) = c(q)$ and $\gamma(q,K) = 0$. That version dichotomizes: (20) determines a path for q, while (21)-(23) determine paths for (K_{+1}, H, X), independently. Hence, the path for M affects q but not (K_{+1}, H, X).[14] When the dichotomy prevails, many properties of this model are similar to one without capital, like LW. Thus, assuming a unique steady state, $\partial q/\partial i < 0$. Since $q < q^*$ for $i > 0$, welfare is maximized at the Friedman Rule $i = 0$. However, if $\theta < 1$, then $q < q^*$ even at $i = 0$. LW interpret this as a holdup problem with money demand: the buyer bears the cost of acquiring cash in the CM, but if $\theta < 1$ he must share the surplus that this cash generates in trade with the seller, which lowers the demand for money and hence q.[15]

The dichotomy does not hold when capital enters the DM cost function, since then K and q both appear in (20) and (21), so there is no way in general to solve for q independently of the other variables. In this case private investors, like the planner, not only take into account the fact that K affects productivity in the CM, but also in the DM (as we will see, however, this does not mean that investment is efficient – i.e. that they take it into account in the same way). A change in monetary policy, by affecting q, thus affects investment. Intuitively,

[14] Although money does not affect CM consumption or investment at the individual level, it does affect individual employment. But the effects cancel so that it does not affect aggregate employment, and agents do not care about effects on individual employment because utility is linear in h (see Aruoba and Wright 2003 for details). Money does affect welfare, however, since it affects q – i.e., dichotomy does not mean (super) neutrality.

[15] See Rocheteau and Waller (2005) for more discussion.

inflation reduces the return to trading in the DM, which affects the value of capital in that market. Since the capital is also used in the CM, this will impact on productivity, employment, output, and consumption in that market.

Notice, however, that even when K enters the DM production function, if $\theta = 1$ then $\gamma(q, K) = 0$. In this case the model is recursive, if not dichotomous: (21)-(23) can be solved for (K_{+1}, H, X), and then, given K, (20) determines q. So when $\theta = 1$, anything like fiscal policy that influences K will affect q, but there is no feedback in the other direction, and monetary policy still cannot influence investment, employment or consumption in the CM. Intuitively, when $\theta = 1$ sellers get none of the DM surplus, so they realize no cost savings from bringing extra capital to the DM and hence the investment decision is based solely on returns in the CM. This holdup problem in the demand for capital, and hence underinvestment, does not require $\theta = 1$. In fact it applies whenever $\theta > 0$, since sellers will always underinvest unless they get the full return.[16]

The distortion described above is in addition to the usual inefficiencies that arise when $i > 0$ in monetary economies, the problem in money demand that arises with bargaining when $\theta < 1$, and the obvious problems associated with distorting taxes. If we ran the Friedman Rule ($i = 0$) and used lump sum taxes exclusively, we would be left with only the holdup problems. In some models all such problems can be resolved simultaneously if one simply sets θ correctly; see Hosios (1990) or, for a recent update, Rogerson et al. (2005). This is impossible here: $\theta = 1$ resolves the problem in the demand for money, but this is the worst case for investment; and $\theta = 0$ resolves the problem in the demand for capital, but this this is the worst case for money (it implies $q = 0$). There is simply no θ that eliminates both problems.

[16] Holdup problems in investment in general are standard fare in microeconomics, but perhaps need more attention in macro. As Caballero and Hamour (1998) put it, "From a macroeconomic perspective, the prevalence of unprotected specific rents makes it a potentially central factor in determining the functioning of the aggregate economy." See also Caballero (1999), who says "the quintessential problem of investment is that is almost always sunk ... opening a vulnerable flank ... The problem is far more serious ... when the exposed flanks are largely controlled by economic agents with the will and freedom to behave opportunistically." Holdup problems are usually attributed to a lack of complete contracting, which makes perfect sense in search-based models, to the extent that it is not possible to contract before you contact someone.

2.2 Price Taking

With care, competitive price taking instead of bargaining can be used in models like this (Rocheteau and Wright 2005). The CM is completely unchanged, while the value function for the DM has the same form as (5), but (6) and (7) change. For a buyer we have

$$V^b(m,k) = \max_q \ \{u(q) + W(m - \tilde{p}q, k)\} \text{ s.t. } \tilde{p}q \leq m, \tag{26}$$

where $\tilde{p}$ is the price level in the DM, now taken parametrically, and for a seller

$$V^s(m,k) = \max_q \{-c(q,k) + W[m + (1-t_q)\tilde{p}q, k]\}. \tag{27}$$

In equilibrium buyers and sellers choose the same q.[17] As in the bargaining version, $d = \tilde{p}q = m = M$ in any equilibrium, and so $q = M/\tilde{p}$. Inserting this into the FOC from (27) and rearranging, we have

$$c_q(q,k) = \frac{(1-t_q)AM}{pqw(1-t_h)}. \tag{28}$$

Given this, the analogs to (18) and (19) from the bargaining model are:

$$\begin{aligned} V_m(m,k) &= \frac{(1-\sigma)A}{pw(1-t_h)} + \frac{\sigma u'(q)}{\tilde{p}} \\ V_k(m,k) &= \frac{A + A(r-\delta)(1-t_k)}{w(1-t_h)} - \sigma c_k(q,k) \end{aligned}$$

Inserting these into (2) yields the analogs to (20) and (21):

$$\frac{c_q(q,K)q}{M} = \frac{\beta c_q(q_{+1},K_{+1})q_{+1}}{M_{+1}}\left[1 - \sigma + \sigma\frac{(1-t_q)u'(q_{+1})}{c_q(q_{+1},K_{+1})}\right] \tag{29}$$

$$\begin{aligned} U'(X) &= \beta U'(X_{+1})\{1 + [F_K(K_{+1},H_{+1}) - \delta](1-t_k)\} \\ &\quad -\sigma\beta(1+t_x)c_k(q_{+1},K_{+1}) \end{aligned} \tag{30}$$

The other equilibrium conditions do not change, and are repeated here for convenience:

$$U'(X) = \frac{A(1+t_x)}{F_H(K,H)(1-t_h)} \tag{31}$$

$$X + G = F(K,H) + (1-\delta)K - K_{+1}. \tag{32}$$

[17] This is the market clearing condition because we have assumed the same measure σ of buyers and sellers, which is convenient but not at all necessary.

Equilibrium is now given by (positive, bounded) paths for (q, K_{+1}, H, X) satisfying (29)-(32), given policy and K_0.

The difference between the bargaining and competitive pricing models is in the difference between (20)-(21) and (29)-(30). They differ because $g(q,K) \neq c_q(q,K)q$ and $g_q(q,K) \neq c_q(q,K)$ in the first pair of equations and $\gamma(q,K) \neq c_k(q,K)$ in the second. Considering steady states, the analogs to (24) and (25) are:

$$\frac{i}{\sigma} = \frac{(1-t_q)\,u'(q)}{c_q(q,K)} - 1 \tag{33}$$

$$\rho = [F_K(K,H) - \delta]\,(1-t_k) - \sigma\,(1+t_x)\,\frac{c_k(q,K)}{U'(X)} \tag{34}$$

Notice (33) and (24) are the same iff $\theta = 1$, while (34) and (25) are the same iff $\theta = 0$. Thus, competitive pricing eradicates the holdup problem in money demand *and* investment. The only distortions remaining are due to taxation and the standard wedge associated with discounting. So if we use lump sum taxes and the Friedman rule, we get efficiency.

3 Extensions and Examples

In this section we discuss an example, and present some extensions designed to show that there are other ways to break the dichotomy.

3.1 Example

Consider the following functional forms, which are the ones we ultimately calibrate below:[18]

$$\begin{aligned} U(x) &= B\frac{x^{1-\varepsilon}-1}{1-\varepsilon} \\ u(q) &= \frac{(q+b)^{1-\eta}-1}{1-\eta} \\ F(K,H) &= K^{\alpha}H^{1-\alpha} \\ c(q,k) &= q^{\psi}k^{1-\psi} \end{aligned} \tag{35}$$

[18]The cost function comes from $\ell(e) = e$ and $q = e^{\chi}k^{1-\chi}$ where $0 < \chi \leq 1$, so $\psi = 1/\chi \geq 1$ (when $\psi = 1$ the model dichotomizes). The other parameters satisfy B, ε, η, $b > 0$ and $0 < \alpha < 1$. The only nonstandard parameter is b; its role is to guarantee $u(0) = 0$ for all η, but we actually set $b \approx 0$, so u is approximately CRRA.

For ease of presentation, for now we focus on pricing taking, and briefly mention bargaining at the end. Then (29)-(32) can be written:

$$\frac{K^{1-\psi}}{q^{-\psi}} = \frac{\beta}{1+\tau}\left[(1-\sigma)\frac{K_{+1}^{1-\psi}}{q_{+1}^{-\psi}} + \sigma(1-t_q)\psi(q_{+1}+b)^{-\eta}q_{+1}\right] \tag{36}$$

$$\frac{X_{+1}^{\varepsilon}}{X^{\varepsilon}} = \beta(1-t_k)\left[\alpha\left(\frac{K_{+1}}{H_{+1}}\right)^{\alpha-1} + 1 - \delta\right] - \frac{\sigma\beta(1+\tau_x)(1-\psi)}{B}\frac{X_{+1}^{\varepsilon}K_{+1}^{-\psi}}{q_{+1}^{-\psi}} \tag{37}$$

$$X = \left[\frac{B(1-\alpha)(1-t_h)}{A(1+t_x)}\frac{K^{\alpha}}{H^{\alpha}}\right]^{1/\varepsilon} \tag{38}$$

$$X = K^{\alpha}H^{1-\alpha} + (1-\delta)K - K_{+1} - G \tag{39}$$

We now show how to solve for a steady state. First let $\mathbb{k} = K/H$, and combine (39) and (38) to get

$$\frac{\mathbb{k}}{K}\left[\frac{(1-\alpha)(1-t_h)}{A(1+t_x)}\mathbb{k}^{\alpha}\right]^{1/\varepsilon} = \mathbb{k}^{\alpha} + (1-\delta)\mathbb{k} + \mathbb{k}_{+1} - \frac{G}{K}\mathbb{k}.$$

In steady state

$$K = \frac{\mathbb{k}^{1-\alpha}\left[\frac{(1-\alpha)(1-t_h)}{A(1+t_x)}B\mathbb{k}^{\alpha}\right]^{1/\varepsilon}}{1-(\delta+\frac{G}{K})\mathbb{k}^{1-\alpha}}. \tag{40}$$

Given $b \approx 0$, (36)-(38) reduce to:

$$q = \left[\frac{\sigma(1-t_q)}{\psi(i+\sigma)}\right]^{\frac{1}{\psi+\eta-1}} K^{\frac{\psi-1}{\psi+\eta-1}} \tag{41}$$

$$X = \left[\frac{(1-\alpha)(1-t_h)B}{A(1+t_x)}\mathbb{k}^{\alpha}\right]^{1/\varepsilon} \tag{42}$$

$$1 = \beta\left[1 + (\alpha\mathbb{k}^{\alpha-1} - \delta)(1-t_k)\right] \tag{43}$$
$$+\frac{(\psi-1)\sigma\beta(1-\alpha)(1-t_h)}{A}\left[\frac{\sigma(1-t_q)}{\psi(i+\sigma)}\right]^{\frac{\psi}{\psi+\eta-1}}\mathbb{k}^{\frac{\alpha(\psi+\eta-1)-(1-\alpha)\psi\eta}{\psi+\eta-1}}\left\{\frac{1-(\delta+G/K)\mathbb{k}^{1-\alpha}}{\left[\frac{(1-\alpha)(1-t_h)B}{A(1+t_x)}\mathbb{k}\right]^{1/\varepsilon}}\right\}^{\frac{\psi\eta}{\psi+\eta-1}}$$

Notice that (43) is one equation in $\mathbb{k}$. The RHS approaches ∞ as $\mathbb{k} \to 0$ and approaches a value less than 1 as $\mathbb{k} \to [1-(\delta+G/K)]^{1/(1-\alpha)}$. Hence it has a solution. The solution is unique if we assume $\alpha(\psi+\eta-1) < (1-\alpha)\psi\eta$, since then the RHS is strictly decreasing. Given $\mathbb{k}$, (40) yields K, (41) yields q, (42) yields X, and $H = \mathbb{k}/K$. So we have existence, uniqueness under a simple condition, and an easy solution method, for steady state.[19]

[19] Things are slightly harder when we use bargaining instead of price taking, since we cannot get a closed form for q as a function of $\mathbb{k}$, but it is still easy to solve numerically.

3.2 Extension: Two Capital Goods

So far, the same stock of capital k was an input to both CM and DM production. Although this is convenient, we now show that the model can accommodate two types of capital, say k in the CM and z in the DM, which depreciate at potentially different rates, say δ and ω. Although they are used as inputs in the different markets, production of both capital goods occurs in the CM in this extension, and following the approach in the baseline model, neither k nor z can be used as a medium of exchange. Also, for the sake of example, there is no tax on z. For this extension, we consider the bargaining version (the price-taking version is similar).

The problem in the CM is now

$$\begin{aligned} W(m,k,z) &= \max_{x,h,m_{+1},k_{+1},z_{+1}} \{U(x) - Ah + \beta V(m_{+1},k_{+1},z_{+1})\} \\ \text{s.t. } (1+t_x)\,x &= w\,(1-t_h)\,h + [1+(r-\delta)\,(1-t_k)]\,k - k_{+1} - T + \frac{m-m_{+1}}{p} \\ &\quad + (1-\omega)\,z - z_{+1}. \end{aligned}$$

The FOC are:

$$\begin{aligned} x &: U'(x) = \frac{A\,(1+t_x)}{w\,(1-t_h)} \\ m_{+1} &: \frac{A\,(1+t_x)}{pw\,(1-t_h)} = \beta V_m(m_{+1},k_{+1},z_{+1}) \\ k_{+1} &: \frac{A}{w\,(1-t_h)} = \beta V_k(m_{+1},k_{+1},z_{+1}) \\ z_{+1} &: \frac{A}{w\,(1-t_h)} = \beta V_z(m_{+1},k_{+1},z_{+1}). \end{aligned}$$

The envelope conditions for W_m, W_k and W_z are derived in the obvious way. The usual logic implies the distribution of (m,k,z) is degenerate for agents leaving the CM.

The DM is as before, except we replace $c(q,k)$ with $c(q,z)$ and $g(q,k)$ with $g(q,z)$. The value function in the DM and the envelope conditions for V_m, V_k and V_z are derived in the obvious way. This leads to:

$$
\begin{aligned}
g(q,Z) &= \frac{\beta g(q_{+1},Z_{+1})}{1+\tau}\left[1-\sigma+\sigma\frac{u'(q_{+1})}{g_q(q_{+1},Z_{+1})}\right] && (44)\\
U'(X) &= \beta U'(X_{+1})\left\{1+\left[F_K(K_{+1},H_{+1})-\delta\right](1-t_k)\right\} && (45)\\
U'(X) &= \beta U'(X_{+1})\left[1-\omega-\frac{(1+t_x)\,\sigma\gamma(q_{+1},Z_{+1})}{U'(x_{+1})}\right] && (46)\\
U'(X) &= \frac{A\,(1+t_x)}{F_H(K,H)\,(1-t_h)} && (47)\\
X+G &= F(K,H)+(1-\delta)\,K-K_{+1}+(1-\omega)\,Z-Z_{+1} && (48)
\end{aligned}
$$

An equilibrium is given by (positive, bounded) paths for $(q, K_{+1}, Z_{+1}, H, X)$ satisfying (44)-(48). Notice (44) is equivalent to (20) with Z replacing K. Also, (45) is the standard condition for K from the one-sector growth model: in contrast to (21) it is not augmented by γ, which now shows up in (46).

This model does not dichotomize, because Z is used in the DM and produced in the CM. An increase in i thus affects Z, and this must affect X, H or K. Notice that for $\theta = 1$, $g_q(q,z) = c_q(q,z)$ and $\gamma(q,z) = 0$. In this case, $i = 0$ generates the efficient q *conditional* on Z. However, when $\theta = 1$ we actually have $Z = 0$, since sellers get no surplus in the DM.[20] In any case, it is clear that the two-capital-good version is similar to the baseline model, and since the latter is easier to analyze and quantify, we revert to it in what follows.

3.3 Extension: Capital Produced in DM

So far, all investment occurs in the CM. Consider the alternative, where k is acquired in the DM, so money is needed to pay it. It has been known since Stockman (1981) that it makes a difference in some models if cash is needed to buy capital. For the sake of illustration, assume agents do not consume the output of the DM at all, but only use it as an intermediate input that is transformed one-for-one into k for production in the CM (Shi 1999 discusses a related model). Assume that each period a fraction σ of agents can produce this intermediate input, the same fraction can transform it into capital, and $1 - 2\sigma$ can do neither.

[20] This extreme form of the holdup problem did not arise in the benchmark model because the same K was used in both markets.

The CM problem is now

$$\begin{aligned} W(m,k) &= \max_{x,h,m_{+1}} \{U(x) - Ah + \beta V(m_{+1},k)\} \\ \text{s.t. } (1+t_x)\,x &= w\,(1-t_h)\,h + [1+(r-\delta)\,(1-t_k)]\,k - T + \frac{m-m_{+1}}{p}. \end{aligned}$$

The FOC are

$$x \quad : \quad U'(x) = \frac{A\,(1+t_x)}{w\,(1-t_h)} \tag{49}$$

$$m_{+1} \quad : \quad \frac{A}{pw\,(1-t_h)} = \beta V_m(m_{+1},k). \tag{50}$$

The envelope conditions are still given by (3) and (4). Since k is obtained in the DM, there is a distribution of k across agents, say $\Phi_k(k)$. Since the FOC for m_{+1} is not independent of k it is not obvious if the distribution of m_{+1} is degenerate; we now show that it is.

Assuming bargaining, the buyer gives up d units of money and acquires q units of intermediate goods which yields $k - k_{-1} = q$ additional units of capital for the CM. The usual methods imply (q,d) is independent of (m_s,k_b,k_s), $d = m_b$, and

$$\frac{m_b}{p} = g(q) = \frac{[\theta c(q) + (1-\theta)qc'(q)]\,[1+(r-\delta)\,(1-t_k)]}{\theta\,[1+(r-\delta)\,(1-t_k)]\,A/w + (1-\theta)c'(q)}.$$

Also,

$$\begin{aligned} V(m,k) &= W(m,k) + \sigma\left[\frac{A + A\,(r-\delta)\,(1-t_k)}{w\,(1-t_h)}q\,(m) - \frac{Am}{pw\,(1-t_h)}\right] \\ &\quad + \sigma\int\left\{\frac{\tilde{d}\,(\tilde{m})\,A}{pw\,(1-t_h)} - c\,[q\,(\tilde{m})]\right\}d\Phi_m(\tilde{m}), \end{aligned}$$

where we integrate wrt to the marginal distribution of m, Φ_m. Hence

$$V_m(m,k) = \frac{A}{pw\,(1-t_h)}\left[1 - \sigma + \sigma\frac{1+(r-\delta)\,(1-t_k)}{g'(q)}\right].$$

Since $V_m(m,k)$ is independent of the buyer's k, the choice of m_{+1} in the CM is the same for everyone by (50). Again, Φ_m is degenerate, whether or not Φ_k is.[21]

[21] As always, we do require interior solutions.

Following the usual procedure, we arrive at:

$$
\begin{aligned}
\frac{g(q)}{F_H(K,H)} &= \frac{\beta g(q_{+1})}{(1+\tau)\,F_H(K_{+1},H_{+1})}\left\{1-\sigma+\sigma\frac{1+[F_K(K_{+1},H_{+1})-\delta]\,(1-t_k)}{g_q(q_{+1},r_{+1},w_{+1})}\right\} && (51)\\
K_{+1} &= (K+\sigma q)(1-\delta) && (52)\\
U'(X) &= \frac{A(1+t_x)}{F_H(K,H)(1-t_h)} && (53)\\
X+G &= F(K,H)+(K+\sigma q)(1-\delta) && (54)
\end{aligned}
$$

An equilibrium is given by (positive, bounded) paths for (q, K_{+1}, H, X) satisfying (44)-(48). This system does not dichotomize, since H and K appear in (51). Intuitively, changing the return on money affects the amount of intermediate goods traded in the DM, and hence K, similar to Stockman (1981). Although this is interesting, we revert to the baseline model in what follows.[22]

3.4 Extension: Nonseparable Utility

Finally, we show how to break the dichotomy with a more general (but still quasi-linear) utility function, $\hat{U}(x,q,e) - Ah$. Although one can do it in a variety of ways, suppose here that x interacts with the (q,e) brought in from the *previous* DM, so the latter are state variables in the current CM. For simplicity and to isolate the effects of nonseparable utility, we assume k does not appear in the DM technology, so that $e = \xi(q) \equiv f^{-1}(q)$.

Now the CM problem is

$$
\begin{aligned}
W(m,k,q,e) &= \max_{x,h,m_{+1},k_{+1}} \left\{\hat{U}(x,q,e) - Ah + \beta V(m_{+1},k_{+1})\right\}\\
\text{s.t. } x &= wh + (1+r-\delta)k - k_{+1} - T + \frac{m-m_{+1}}{p},
\end{aligned}
$$

[22]Implicitly we did not allow existing k to trade in the CM in this version of the model, but we now argue that this is without loss of generality. Suppose agents can trade existing k, and let λ be the price. The FOC for k_{+1} is $A\lambda/w\,(1-t_h) = \beta V_k(m_{+1},k_{+1})$. Inserting $V_k = W_k$ leads to

$$
\frac{\lambda}{F_H(K,H)} = \frac{\beta\lambda_{+1}\left\{1+[F_k(K_{+1},H_{+1})-\delta]\,(1-t_k)\right\}}{F_H(K_{+1},H_{+1})}.
$$

This is independent of individual k, and merely pins down the path for λ in the secondary market so that no arbitrage opportunities exist. Agents are indifferent to trading capital at this price, so the distribution Φ_k is not pinned down. Intuitively, this is because payoffs are linear in wealth (again we require interiority for the result).

where we shut down distorting taxes to keep the notation manageable. The FOC are:

$$\begin{aligned} x &: \hat{U}_x(x,q,e) = \frac{A}{w} \\ m_{+1} &: \frac{A}{pw} = \beta V_m(m_{+1},k_{+1}) \\ k_{+1} &: \frac{A}{w} = \beta V_k(m_{+1},k_{+1}) \end{aligned} \tag{55}$$

We again get a degenerate distribution of (m,k), but now there is a distribution of x in the CM, since this choice for an agent is affected by what happened in the previous DM. Let $x_s = x_s(q,w)$, $x_b = x_b(q,w)$ and $x_0 = x_0(w)$ be the choices of agents who were sellers, buyers and non-traders in the previous DM, where from the first condition in (55)

$$\hat{U}_x[x_s,0,\xi(q)] = \hat{U}_x(x_b,q,0) = \hat{U}_x(x_0,0,0) = \frac{A}{w}.$$

Assuming bargaining, by the usual logic $d = m_b$ and q solves the FOC from the Nash maximization problem:

$$\begin{aligned} 0 &= \theta\left[W(m_s+d,k_s,0,e) - W(m_s,k_s,0,0)\right]\hat{U}_q(x_b,q,0) + \\ &\quad +(1-\theta)\left[W(m_b-d,k_b,q,0) - W(m_b,k_b,0,0)\right]\hat{U}_e\left[x_s,0,\xi(q)\right]\xi'(q) \end{aligned} \tag{56}$$

We can rearrange (56) as

$$\frac{m_b}{p} = \frac{g(q,w)w}{A},$$

which is similar in spirit to (14), except that now $g(q,w)$ solves

$$\begin{aligned} \Upsilon(q,w)g(q,w) &= (1-\theta)\left\{\hat{U}\left[x_0(w),0,0\right] - \hat{U}\left[x_b(q,w),q,0\right]\right\}\hat{U}_e\left[x_s(q,w),0,\xi(q)\right]\xi'(q) \\ &\quad +\theta\left\{\hat{U}\left[x_0(w),0,0\right] - \hat{U}\left[x_s(q,w),0,\xi(q)\right]\right\}\hat{U}_q\left[x_b(q,w),q,0\right] \\ &\quad +(1-\theta)\frac{A}{w}\left[x_b(q,w) - x_0(w)\right]\hat{U}_e\left[x_s(q,w),0,\xi(q)\right]\xi'(q) \\ &\quad +\theta\frac{A}{w}\left[x_s(q,w) - x_0(w)\right]\hat{U}_q\left[x_b(q,w),q,0\right] \end{aligned} \tag{57}$$

with

$$\Upsilon(q,w) \equiv \theta U_q\left[x_b(q,w),q,0\right] - (1-\theta)U_e\left[x_s(q,w),0,\xi(q)\right]\xi'(q) > 0. \tag{58}$$

This may appear onerous, but it simplifies a lot in some special cases. Of course, if $\hat{U} = U(x) + u(q) - \ell(e)$ is separable, then $g(q, w) = g(q)$ from LW, and we are back to a model that dichotomizes. Consider the intermediate case $\hat{U} = \tilde{U}(x, q) - \ell(e)$, where we can again write $c(q) = \ell[\xi(q)]$ because e and q enter separably. Then the RHS of (57) reduces to

$$\theta c(q)\tilde{U}_q\left[x_b(q,w),q\right] + (1-\theta)\left\{\tilde{U}\left[x_b(q,w),q\right] - \tilde{U}\left[x_0(w),0\right] + \frac{A}{w}\left[x_0(w) - x_b(q,w)\right]\right\}$$

while $\Upsilon(q, w) = \theta\tilde{U}_q\left[x_b(q,w),q\right] - (1-\theta)\, c'(q)$. This case is quite easy to analyze. Moreover, for any $\hat{U}$, if $\theta = 1$ then

$$g(q,w) = U\left[x_0(w),0,0\right] - U\left[x_s(q,w),0,\xi(q)\right] + \frac{A}{w}\left[x_s(q,w) - x_0(w)\right]. \tag{59}$$

In any of these cases, the usual methods lead to the equilibrium condition

$$g(q,w) = \frac{\beta g(q_{+1}, w_{+1})}{1+\tau}\left[1 - \sigma + \sigma\frac{U_q\left[x_b(q_{+1},w_{+1}),q_{+1},0\right]}{g_q(q_{+1},w_{+1})}\right].$$

It is clear that q cannot be determined independently of $w = F_H(K, H)$, in general, unless $\hat{U}$ is separable. A steady-state satisfies:

$$\begin{aligned}
\frac{i}{\sigma} &= \frac{U_q\left\{x_b\left[q, F_H(K,H)\right], q, 0\right\}}{g_q\left[q, F_H(K,H)\right]} - 1 \\
\rho &= F_K(K,H) - \delta \\
X &= F(K,H) - \delta K
\end{aligned}$$

Here, aggregate CM consumption and employment are given by

$$\begin{aligned}
X &= \sigma x_b\left(q,w\right) + \sigma x_s\left(q,w\right) + \left(1-2\sigma\right)x_0\left(w\right) \\
H &= \sigma h_s + \sigma h_b + \left(1-2\sigma\right)h_0,
\end{aligned}$$

with individual x_j given by (59), and then individual h_j given by the budget equation.

This model is not especially hard to solve, and it can be interesting for some applications (see e.g. Rocheteau et al. 2005). However, for most of what follows we will return to the case of $\hat{U} = U(x) + u(q) - \ell(e)$, and pursue the impact of breaking the dichotomy by having capital enter the DM technology in the baseline model.

4 Quantitative Analysis

The next step is to quantify the model. For both bargaining and price taking, we will calibrate the baseline model, solve for the decision rules, and discuss some policy experiments. But first, we need to do some simple accounting.

The price levels in the DM and CM are $\tilde{p} = M/q$ and p, where p satisfies

$$p = \frac{AM}{(1 - t_h)\, g(q, K)\, w} \tag{60}$$

in the bargaining version by (14), and

$$p = \frac{A(1 - t_q)\, M}{(1 - t_h)\, qc_q(q, K)\, w} \tag{61}$$

in the price-taking version of the model by (28). Nominal outputs of the two markets are σM and $pF(K, H)$. We use p as the unit of account by which we convert all nominal variables into real terms. Hence, real GDP is

$$Y = \sigma \frac{M}{p} + F(K, H). \tag{62}$$

The share of output produced in the DM is $s_D = \sigma M / pY$.

Define the markup μ by equating $1+\mu$ to the ratio of price to marginal cost. The markup in the CM market is always 0, since it is competitive. The markup in the DM under price taking is also 0. With bargaining, however, the markup in the DM is derived as follows. Marginal cost in terms of utility is $c_q(q, K)$. Due to quasi-linearity, a dollar is always worth $A/p(1 - t_h)\, w$ utils, so marginal cost in dollars is $c_q(q, K)\, p(1 - t_h)\, w/A$. The (after-tax) price in dollars in the DM is $(1 - t_q)\, \tilde{p} = (1 - t_q)\, M/q$. The DC markup μ_D is therefore given by

$$1 + \mu_D = \frac{(1 - t_q)\, M/q}{c_q(q, K)\, p(1 - t_h)\, w/A} = \frac{(1 - t_q)\, g(q, K)}{qc_q(q, K)}, \tag{63}$$

after eliminating M. Hence, the aggregate markup is $\mu = s_D \mu_D$, where s_D is the share of the DM defined above.

4.1 Steady State Calibration

We now describe our calibration strategy, using the functional forms from Section 3. Several parameters can be set immediately. First, we set the discount factor β to match a real interest in the data of $r = 0.035$.[23] Now recall

$$U(x) = B\frac{x^{1-\varepsilon} - 1}{1-\varepsilon} \text{ and } u(q) = \frac{(q+b)^{1-\eta} - 1}{1-\eta}.$$

We set $b = 0.0001$, so that the utility of consumption in the DM is approximately CRRA, as it is in the CM. As a benchmark we set $\varepsilon = \eta = 1$, mainly to facilitate comparison with previous studies, but we check the robustness of the conclusions to this choice below.[24] Then we are left with the weight on CM consumption B, as well as the constant on CM hours A, which are set along with several other parameters as described below.

Moving to policy parameters, we can directly observe the average inflation rate $\tau = 0.036$. We can also directly observe taxes,

$$(t_h, t_k, t_x, t_q) = (0.242, 0.548, 0.069, 0.065),$$

where we use the measures of average effective marginal rates from McGrattan et al. (1997) for t_h and t_k, and the average of excise plus sales tax over consumption expenditure which is equal to t_x in the centralized market and $t_q/(1-t_q)$. The other policy parameter is G. Although we can observe G/Y in the data, since Y is endogenous, we actually solve for G and several other parameters simultaneously as described below.

Moving on to technology, we set depreciation $\delta = 0.070$ to match observed I/K, where K includes residential and nonresidential structures, plus producer equipment and software (but not consumer durables or inventories). Finally, we have two parameters α and ψ from the CM and DM technologies, $F(K,H) = K^{\alpha}H^{1-\alpha}$ and $c(q,k) = q^{\psi}k^{1-\psi}$, the probability

[23] When we mention the data, we mean annual U.S. data from 1951-2004. Our real interest rate is computed from an average nominal rate on Aaa-rated corportate bonds of 7.2% and an average inflation rate (changes in the GDP deflator) of 3.6% over this period.

[24] Also, $\varepsilon = \eta = 1$ is a good benchmark because this is what we require for balanced growth in a generalized version of the model with technical change (details available upon request).

of being a consumer or a producer in the DM σ, and (in the model with bargaining) the bargaining power parameter θ. Table 1 summarizes the seven parameters left to determine, along with seven observations that we now discuss.

Parameters	A	B	G	α	ψ	σ	θ
Observations	H	v	G/Y	LS	K/Y	ϵ	μ

Table 1: Calibration Parameters and Targets

First, we have average hours worked as a function of discretionary time, which as is standard we set to $H = 1/3$ (see e.g. Juster and Stafford 1991). Second we have average velocity, which is $v = M/PY = 5.76$ in our data when we measure M by $M1$.[25] Third we have $G/Y = 0.251$. Fourth we have $K/Y = 2.319$, given our choice for K described above. Fifth we have labor's share of income, which we measure to be $LS = 0.712$. Sixth we have the elasticity of M/p wrt i, which we estimate to be $\epsilon = -0.226$, as discussed below. Finally, we have the markup, which we set to $\mu = 0.10$ (see e.g. Basu and Fernald 1997). We choose parameters to minimize the distance between these targets in the data and the model (except we ignore the markup when we assume price-taking). The numerical values that result from this procedure are reported below for various versions of the model.

Before discussing the results, we mention that these measurements are all fairly standard, with a couple of exceptions that are worth comment. First, we use *both* K/Y and LS as targets. In a typical application, once one sets the share parameter in the CM technology α, given δ, ρ and taxes, one knows K/Y – but this is not true here because we also have the DM technology. The idea is that α is used to match LS and then ψ is used to match K/Y. This works well here because, given α matches LS, K/Y tends to be too low in models with taxes, especially with t_k around 1/2, as it is here (see Greenwood et al. 1995 for more discussion). In this model we can in principle match both, because the returns to capital come, in general, from both the CM and DM technologies.[26]

[25] As mentioned in Section 2, one can recast the model by introducing banks as in He, Huang and Wright (2005), and have agents in the DM pay with either cash or checks or debit cards. This suggests $M1$ is the right empirical notion of money; in any case, we check robustness to using other measures.

[26] The discussion in this paragraph suggests we can set α to match LS and then set ψ to match K/Y

Second, we need to describe how we measure the interest elasticity of money demand ϵ. We specify log real money demand (m_t) as a linear function of log nominal interest rate (i_t) and log real output (y_t) and we allow for first order autocorrelation in the residuals, which is a common specification in the money demand literature (see e.g. Goldfeld and Sichel 1990). Considering the nonstationary feature of the variables involved in the estimation, we estimate this equation in first differences and we get the following estimates:[27]

$$\Delta m_t = \beta_y \Delta y_t + \beta_i \Delta i_t - \rho \beta_y \Delta y_{t-1} - \rho \beta_i \Delta i_{t-1} + \rho \Delta m_{t-1} + \varepsilon_t \tag{64}$$

$$\beta_y = 0.369 \ \ (0.124)\,, \ \beta_i = -0.226 \ \ (0.045)\,, \ \rho = 0.347 \ \ (0.131) \ \ R^2 = 0.423$$

where ρ is the AR(1) coefficient for the residuals in the original equation and the numbers in the parantheses are the standard errors. The estimated long-run interest elasticity is $\epsilon - 0.226$, with a relatively small standard error of 0.05. We match this to the theoretical long-run elasticity from the model, which is easily computed as a function of parameters as a comparative statics exercise.[28]

This completes the discussion of our calibration method (again, the actual numbers for the parameters are reported below). One might like to have some notion of how well the method does, beyond saying that we get close to the targets. We could use formal econometric procedures to provide an answer, of course, and there is no reason not to pursue this line in future work, but it goes beyond the scope of this project. Alternatively, one could follow a standard business cycle approach and simulate the model, hitting it with stochastic shocks as one sees fit, and comparing second moments generated by the model and data. This is on the agenda, but again beyond the scope of this project. For now, we pursue some simpler alternatives.

independently, but this is only meant to be heuristic; we actually solve for the seven parameters to match the seven observations simultaneously.

[27] Note that since we are taking first differences of logs, the equation we estimate is in essence in terms of growth rates. The parameter restrictions in the equation trivially follow from the original model in levels.

[28] Heuristically speaking, σ is the key parameter for pinning down ϵ, and B is the key for pinning down v, as suggested by Table 1; but again, we actually solved for the parameters simultaneously to match all observations.

One thing we can do is to see how well the model matches up with the money demand data, which is a common if somewhat informal way to proceed, used e.g. by Lucas (2000). Figure 1 shows (64) predicts well in a time series sense. Figure 2 shows the relationship on which Lucas focuses, $M/pY = 1/v$ versus i, in the data and in the model, both for the bargaining and the price-taking versions calibrated as described above. Notice that, as is typical, it is not easy to fit the low interest rate observations in the upper left part of the scatter plot. These points are all from 1951-60; ignoring this decade, we think our money demand curve looks pretty good, about as good as those used by Lucas, say.[29]

A related way to ask how well the model does is to look at the relationship between inflation τ and real variables, such as investment – after all, our objective here is to analyze models that do not dichotomize where money affects the CM variables. Using quarterly data, we estimate the long-run elasticity of I to τ to be around -0.02 and significant.[30] This may appear small, but if one thinks about it correctly it is not: doubling the inflation rate from our benchmark value of 0.035 to 0.7 would cause aggregate investment to fall by 2%, which is nothing to scoff at. Given the calibration, we can compute the elasticity implied by the theory. While both the bargaining and price-taking models generate a negative relation between τ and I, in the former the effect is weak (an elasticity of -0.001) and in the latter it is too strong (an elasticity of -0.057).

The reason the bargaining model generates only a weak negative relation between τ and I is the holdup problem. The returns to having more capital in the DM are fairly low because they have to be split between the consumer and producer, and the calibrated value of θ actually gives a fairly large share to the former. So while inflation, as a tax on DM activity, does reduce the incentive to invest, the effect is small. Of course the effect depends on the calibrated curvature in the DM cost function, which comes out around $\psi = 1.67$. But this is the best this model can do on this dimension: If we choose ψ to make the effect as big

[29]It is notoriously well known that money demand is not what one would call stable. The other observations that are hard to fit are those lying close to the horizontal axis; these are all from 1999-2004. It seems clear (and not surprising) that money demand is shifting down and flattening out over time

[30]This estimate is obtained in a similar way as (64). The standard error of the estimate is 0.008.

as possible holding the other parameters fixed, or if we replace K/Y say as a target by the relevant elasticity and recalibrate, we end up at basically the same ψ and elasticity. The holdup problem simply chokes off this effect quantitatively.

In the price-taking version, there is no holdup problem, and the effect of inflation on investment is quite big. Indeed it is too big, at -0.057, but this is not a serious issue: if we want to do better on this dimension, we can replace K/Y or LS as a target by the relevant elasticity, and get it spot on at -0.02 with relatively little sacrifice in terms of other targets. So the price-taking version can do very well on this dimension. But of course, it cannot match the markup. We find this an interesting tradeoff: each version of the model does better on one dimension. What is certainly true, however, is that to the extent that one takes seriously a negative relation between I and τ, this is inconsistent with any model that dichotomizes, such as either version with $\psi = 1$.[31]

4.2 Solving for Decision Rules

The above discussion is about steady states. For much of what we want to do we need to go beyond this and solve for the equilibrium decision rules, because we want to be able to analyze the transition between steady states after a policy change. As is standard, we scale nominal variables by dividing by the aggregate money stock, so that $\hat{m} = m/M$ and $\hat{p} = p/M$, e.g. Then the individual state variable becomes $(\hat{m}, k, K)$. In equilibrium, $\hat{m} = 1$ and $k = K$. A recursive equilibrium is then described by time-invariant functions $[q(K), K_{+1}(K), H(K), X(K)]$, solving (20)-(23) for the bargaining version and (29)-32) for the price-taking version, plus value functions $[W(K), V(K)]$ solving versions of the Bellman equations (1) and (8). We solve these equations numerically, using the Weighted Residual Method with Chebyshev Polynomials and Orthogonal Collocation. This is a nonlinear and global approximation which will be especially useful for accurate welfare computations.[32]

[31] In the baseline model, it is not possible to generate a *positive* relation between τ and output, investment, consumption or employment in the CM, but one could do so by considering the extension in the previous section to nonseparable utility.

[32] See Judd (1992) for details, and Aruoba et al. (2005) for a recent comparison of different solution methods.

Figure 3 plots the decision rules for a typical parametrization.[33]

For expositional purposes, let $[q^{\pi}(K), K^{\pi}_{+1}(K), H^{\pi}(K), X^{\pi}(K)]$ and $[W^{\pi}(K), V^{\pi}(K)]$ describe equilibrium given government policy π. A steady state solves

$$K^{\pi} = K^{\pi}_{+1}(K^{\pi}) \tag{65}$$

Generally, there is a non-trivial transition path after a change in π. When we compare two policies π_1 and π_2, our welfare comparisons are between $W^{\pi_1}(K^{\pi_1})$ and $W^{\pi_2}(K^{\pi_1})$ – that is, we look at utility under the new policy starting at the old steady state. For computing welfare, we use a standard consumption-equivalent measure: we compute the Δ such that going from policy π_1 to policy π_2 and reducing all (CM and DM) consumption by a factor Δ makes agents just as well off as staying at π_1.

5 Results

5.1 Main Results

Consider Table 2. As a benchmark, the first two columns report results when we give up on the target of LS and simply fix $\psi = 1$, which is the special case in which the model dichotomizes. The first column also gives up on the markup μ and fixes $\theta = 1$, while the second includes μ as a target and calibrates θ. In the first column, with $\theta = 1$, because there is no holdup problem with money demand and the model dichotomizes, the results are identical with price taking (but only because $\psi = 1$). The other columns include ψ as a calibrated parameter. The third column fixes $\theta = 1$, the fourth calibrates θ and targets μ, and the final column uses price taking, again giving up on μ. In each of these cases the model matches well most of the targets, including H, K/Y, G/Y, v, and ϵ.

The bargaining models are not as good on LS, which comes out closer to 0.64 rather than 0.71. This is because with a holdup problem and taxes, it is difficult to match K/Y without having a relatively high α. The price-taking model hits LS and K/Y, but of course

[33] Even though the decision rules look roughly linear, there is actually some subtle nonlinearity in the decision rules.

misses the markup μ. When we do calibrate to the markup μ in the bargaining versions, θ comes out to be around 3/4. Notice that in the third column we actually get a negative markup, which would be strange in competitive model but is no problem in a bargaining model; because $\theta = 1$ in this column, the consumers are making take it or leave it offers, so price equals average cost which exceeds marginal cost.[34] When ψ is calibrated, the value is around 1.7 in the bargaining models and 2.7 under price taking; the data want a model that does not dichotomize.

Some things that we do not calibrate to in any of the cases are also reported. The elasticity of investment with respect to inflation implied by the first three columns is identically 0, since the first two dichotomize, and while in the third case there is feedback from the CM to q there is no feedback from the DM or monetary policy to the CM. We already discussed this elasticity in the other two cases. We also report the share of the DM in output s_D, which varies between 4.2% and 4.7%. We think these numbers are very reasonable, in the sense that we would be uncomfortable if the model predicted that anonymous bilateral trade was too big a share of GDP. Because s_D is relatively small, we need a big markup in the DM to match the average markup. One reason s_D is small is that the probability σ is relatively small, around 1/4.

Table 3 reports for each column ratios of (q, K, H, X, Y) at two values of τ, 10% and the value corresponding to the Friedman rule, where $i = 0$. In the models with $\psi = 1$, this ratio for q is around 0.65, but when capital is used in the DM the ratio is closer to 0.75 in the bargaining models and 0.8 in the competitive model. Thus, inflation does not reduce DM output as much when capital is used to produce it. GDP goes down by around 2% with 10% in all columns but the last, where it actually goes down by over 5%. This is because in this model the calibrated ψ is big – well over 2 – which means capital is important in DM production. Since there is no holdup problem in the competitive economy, agents take this into account when they invest; with inflation the decline in q reduces K by over 10%. This

[34] This is not true in the first column because when $\psi = 1$ average cost equals marginal cost.

has a big effect on CM output.

The table also reports for each column ratios of (q, K, H, X, Y) at the equilibrium at $\tau = 0.1$ and at the first best. Naturally, these numbers are relatively small, due to the large effect of distorting taxation in the model. Thus, in equilibrium Y and X are only 2/3 of their first best values, K and q are around 1/2 of the first best, and H is 3/4 of the first best. Again, these effects are mainly due to distorting taxation – one can calculate how much is due to taxation and how much to inflation by combining the results discussed in this and the previous paragraphs.

We also report the welfare implications of reducing inflation from 10% to the Friedman rule.[35] In the first column, which means a model that dichotomizes and $\theta = 1$ (or equivalently with $\psi = 1$, competitive pricing), this is worth just under 1% of consumption. This is also true in the third column. These results are commensurate with what Lucas (2000) finds, presumably because these models do not have holdup problems. In columns 2 and 4, in bargaining models calibrated to match the markup μ, we find that reducing inflation from 10% to the Friedman rule is worth over 3%. In column 5, which is competitive pricing and $\psi = 2.7$, it is worth closer to 2%, which is still big even though there is no holdup problems because of the way inflation affects K as discussed above.

Only in the last two columns is there an effect of τ on K, and hence only in these cases is there a nontrivial transition after a reduction in inflation. During the transition, welfare goes down with a reduction in τ because agents have to work and save more to build up K, but in the long run they are better off. The table also reports the welfare gain of switching from a equilibrium at $\tau = 0.1$ to the first best, but only in terms of steady states (we do not compute the transition path to the first best because it is not an equilibrium). These numbers are very big, but again this is mainly due to the effects of eliminating distortionary taxation. Similar results appear in nonmonetary models when distortionary taxation is eliminated – e.g. McGrattan et al. (1997) find it is worth around 30% of consumption.

[35] As noted earlier, this is not a steady state comparison but we take the welfare during the transition in to account.

To understand the source of the welfare loss during the transition, consider Figure 4 and Figure 5, where we plot the transition path of the variables of interest following a reduction of the inflation rate from 10% to the Friedman rule in period 1 for the parametrization in column 4 and 5, respectively. Unity on the y-axis corresponds to the initial steady state. All lines converge to the new steady state in about 50 periods for the bargaining version and 70 periods for the price taking version. We see that DM output immediately jumps up by a large amount and quickly converges to its new steady state value. Hours in the CM, on the other hand jump up initially and slowly go down to the new steady state value, just above the original one. Similarly, consumption in the CM jumps down by a small amount on impact before slowly converging to a new level. In order to accumulate the extra capital, agents work more and consume less initially. This is the source of the welfare loss during the transition. Comparing the two figures, it becomes clear that the welfare loss during the transition is larger in the price taking version because the amount of capital that needs to be accumulated in order to reach the new steady state is bigger.

5.2 Robustness

To be added.

6 Conclusions

To be added.

A Appendix

A.1 The Cost Function

Here we verify the properties of the DM (utility) cost function $c(q,k)$ that we stated in Section 2. This cost function comes from a DM production function $q = f(k,e)$ that is strictly increasing and concave, and a disutility of effort function $\ell(e)$ that is is strictly increasing and convex. Also, by definition, saying k is a normal input into f means that in the problem $\min\{we+rk\}$ s.t. $f(k,e) \geq q$, the solution satisfies $\partial k/\partial q = f_e f_{ek} - f_k f_{ee} > 0$.

To proceed, first rewrite $q = f(k,e)$ as $e = \xi(q,k)$. Then $\partial e/\partial q = \xi_q = 1/f_e > 0$ and $\partial e/\partial k = \xi_k = -f_k/f_e < 0$. Also $\xi_{qq} = -f_{ee}/f_e^3 > 0$, $\xi_{kk} = -\left(f_e^2 f_{kk} - 2f_e f_k f_{ke} + f_k^2 f_{ee}\right)/f_e^3 > 0$, and $\xi_{kq} = -\left(f_{ek}f_e - f_{ee}f_k\right)/f_e^3$. Hence, $c_q = \ell'/f_e > 0$, $c_k = -\ell' f_k/f_e < 0$, $c_{qq} = \left[\ell''\ell'^2 f_e - \ell' f_{ee}\right]/f_e^3 > 0$, $c_{kk} = -\left[\ell'\left(f_e f_{kk} - 2f_e f_k f_{ke} + f_k^2 f_{ee}\right) - f_e f_k^2 \ell''\right]/f_e^3 > 0$ and $c_{qk} = -\left[\ell'' f_e f_k - \ell'\left(f_k f_{ee} - f_e f_{ek}\right)\right]/f_e^3$. These results establish that c is increasing and convex in q and decreasing and convex in k, and that $c_{qk} < 0$ if k is a normal input, as asserted in the text.

A.2 Money Demand Elasticity

The interest elasticity of money demand is $\epsilon = \frac{\partial(M/P)}{\partial i}\frac{i}{M/P}$. We show how to compute this in the bargaining model (the price-taking version is similar). First, using (60),

$$\epsilon = \left[g_q \frac{\partial q}{\partial i} + g_k \frac{\partial K}{\partial i}\right]\frac{i}{g} + \left[F_{HH}\frac{\partial H}{\partial i} + F_{HK}\frac{\partial K}{\partial i}\right]\frac{i}{F_H},$$

where we drop the arguments to save space. It is now a matter of substituting $\partial q/\partial i$, $\partial K/\partial i$ and $\partial H/\partial i$.

Eliminating X, we can write the steady state as 3 equations in (q,K,H):

$$\frac{i}{\sigma} = \frac{u'(q)}{g_q(q,K)} - 1 \tag{66}$$

$$\rho = \left[F_K(K,H) - \delta\right](1-t_k) - \frac{\sigma(1+t_x)\gamma(q,K)}{U'\left[F(K,H) - \delta K - G\right]} \tag{67}$$

$$U'\left[F(K,H) - \delta K - G\right]F_H(K,H) = \frac{A(1+t_x)}{(1-t_h)} \tag{68}$$

We take the total derivative of this system to obtain

$$B\begin{bmatrix} dq \\ dK \\ qH \end{bmatrix} = \begin{bmatrix} di \\ 0 \\ 0 \end{bmatrix}$$

where

$$B = \begin{bmatrix} \frac{\sigma(g_q u'' - u' g_{qq})}{g_q^2} & -\frac{\sigma u' g_{qk}}{g_q^2} & 0 \\ -\frac{\sigma(1+t_x)\gamma_q U'}{U'^2} & \Theta & \frac{(1-t_k)U'^2 F_{KH} + \sigma(1+t_x)\gamma U'' F_H}{U'^2} \\ 0 & (F_K - \delta) F_H U'' + F_{KH} U' & F_H^2 U'' + F_{HH} U' \end{bmatrix} \tag{69}$$

and $\Theta = (1 - t_k) F_{KK} - \frac{\sigma(1+t_x)}{(U')^2} [\gamma_k U' - (F_K - \delta) \gamma U'']$. We can now compute the partials as

$$\frac{\partial q}{\partial i} = B_{11}^{-1} \ \frac{\partial K}{\partial i} = B_{21}^{-1} \ \frac{\partial H}{\partial i} = B_{31}^{-1}$$

where B_{ij}^{-1} refers to the (i, j) element of B^{-1}.

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Table 2 - Calibration Results

	(1) AW-B	(2) AW-B	(3) AWW-B	(4) AWW-B	(5) AWW-PT
Calibrated Parameters					
α	0.36	0.36	0.36	0.36	0.29
σ	0.24	0.27	0.24	0.27	0.27
B	2.59	2.57	1.57	1.55	2.63
ψ	**1.00**	**1.00**	1.65	1.67	2.80
A	6.28	6.26	3.80	3.77	7.05
G	0.14	0.14	0.14	0.14	0.13
θ	**1.00**	0.77	**1.00**	0.74	–
Calibration Targets from the Benchmark Economy					
Markup (10.00)	0.00	10.00	-1.65	10.00	0.00
K/Y (2.32)	2.31	2.32	2.31	2.32	2.32
LS (0.71)	0.65	0.64	0.65	0.64	0.71
G/Y (0.25)	0.25	0.25	0.25	0.25	0.25
Velocity (5.76)	5.76	5.76	5.76	5.76	5.76
Int Elast of M (-0.23)	-0.23	-0.23	-0.23	-0.23	-0.23
Miscellaneous					
Share of DM	4.16	4.73	4.17	4.60	4.75
Markup in DM	0.00	211.47	-39.53	217.22	0.00
Inf Elast of I (-0.02)	0.000	0.000	0.000	-0.001	-0.057

Notes: AW refers to the model in Aruoba and Wright (2003), which corresponds to the case where $\psi = 1$. B refers to the bargaining version and PT refers to the price taking version of the model. The first column also corresponds to the price taking case in AW. Bold parameters show restricted parameters.

Table 3 - Allocations and Welfare with 10% Inflation

	(1) AW-B	(2) AW-B	(3) AWW-B	(4) AWW-B	(5) AWW-PT
Compared to the Friedman Rule					
$q(\tau)/q(F)$	0.63	0.65	0.76	0.77	0.80
$y(\tau)/y(F)$	1.00	1.00	1.00	$1.00^{(*)}$	0.96
$Y(\tau)/Y(F)$	0.98	0.98	0.98	0.98	0.94
$K(\tau)/K(F)$	1.00	1.00	1.00	$1.00^{(*)}$	0.89
$H(\tau)/H(F)$	1.00	1.00	1.00	$1.00^{(*)}$	0.99
$X(\tau)/X(F)$	1.00	1.00	1.00	$1.00^{(*)}$	0.97
Compared to the Solution to Planner's Problem					
$q(\tau)/q^*$	0.59	0.19	0.49	0.18	0.54
$y(\tau)/y^*$	0.62	0.62	0.58	0.57	0.67
$Y(\tau)/Y^*$	0.61	0.61	0.55	0.55	0.67
$K(\tau)/K^*$	0.44	0.44	0.37	0.37	0.50
$H(\tau)/H^*$	0.75	0.75	0.73	0.73	0.75
$X(\tau)/X^*$	0.59	0.59	0.56	0.55	0.63
Welfare Gains (%)					
10% to Friedman Rule	0.97	3.21	0.91	3.38	1.83
During Transition	0.00	0.00	0.00	−0.03	−1.64
10% to 0%	0.82	2.28	0.78	2.38	1.31
During Transition	0.00	0.00	0.00	−0.02	−1.09
Steady State Welfare Loss versus Solution to Planner's Problem (%)					
10% Inflation	28.12	37.54	36.75	52.74	22.27
Friedman Rule	26.90	33.26	35.51	47.72	18.17

Notes: The entries with $^{(*)}$ are rounded up and are not identically equal to unity. y refers to the output in the CM.

Figure 1 - Money Demand Estimation

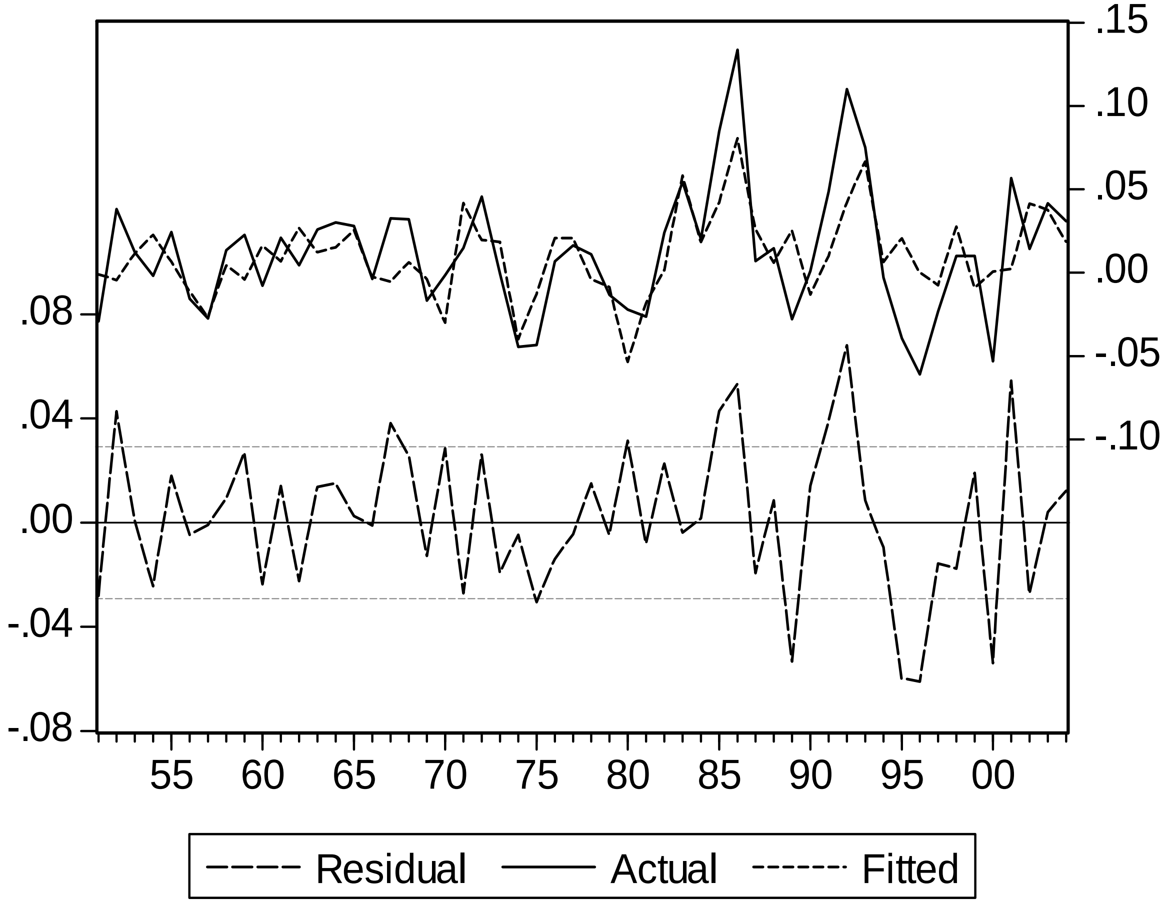
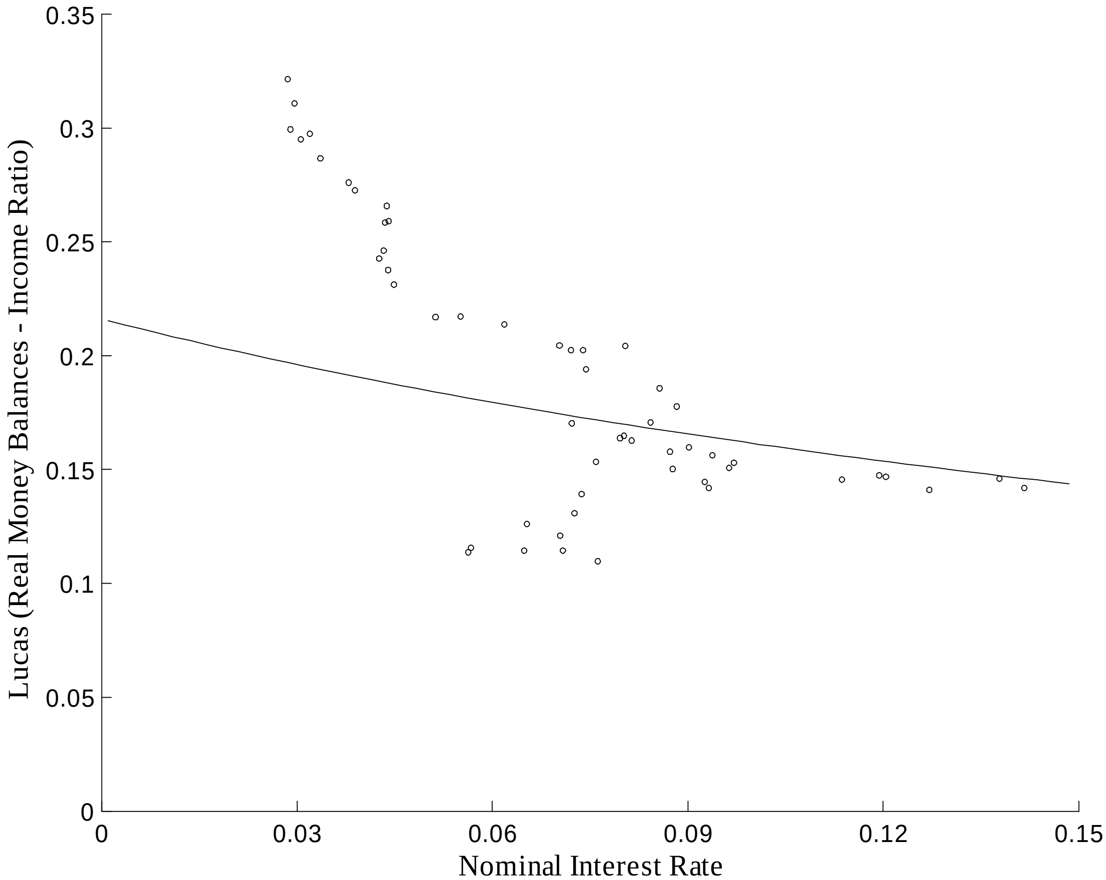
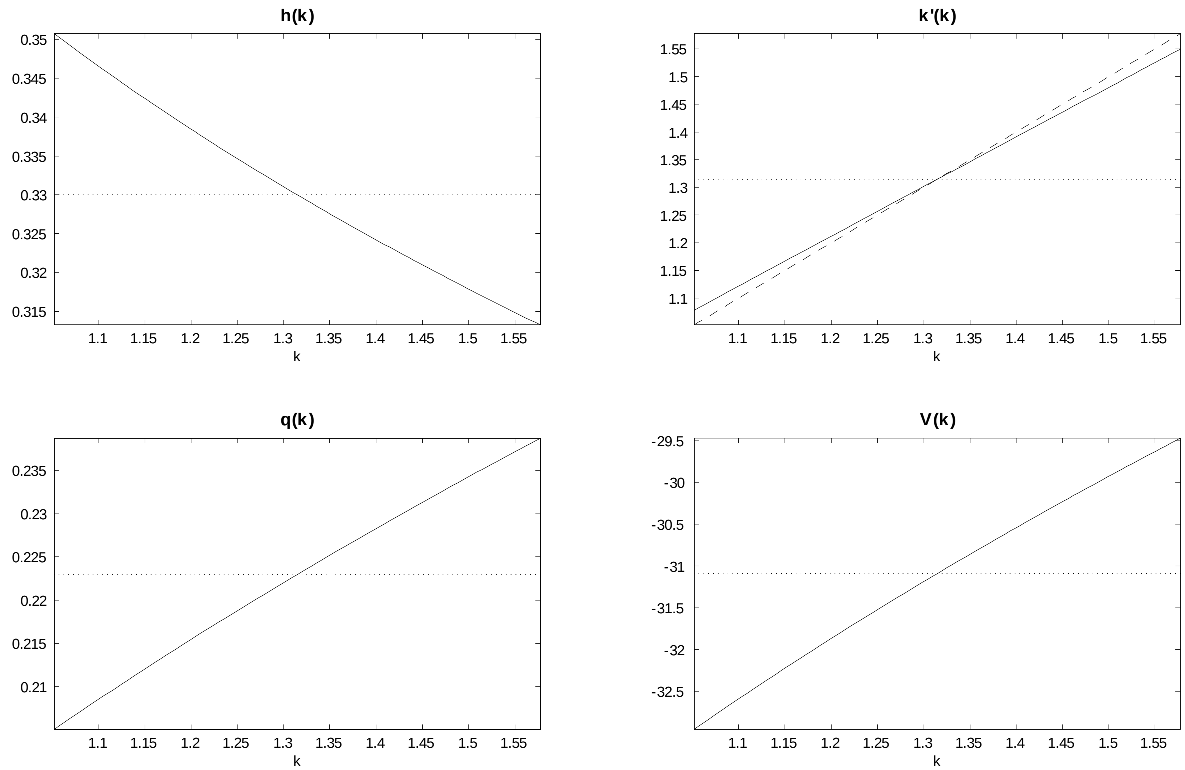


Figure 2 - Fit of Real Money Balances - Income Ratio



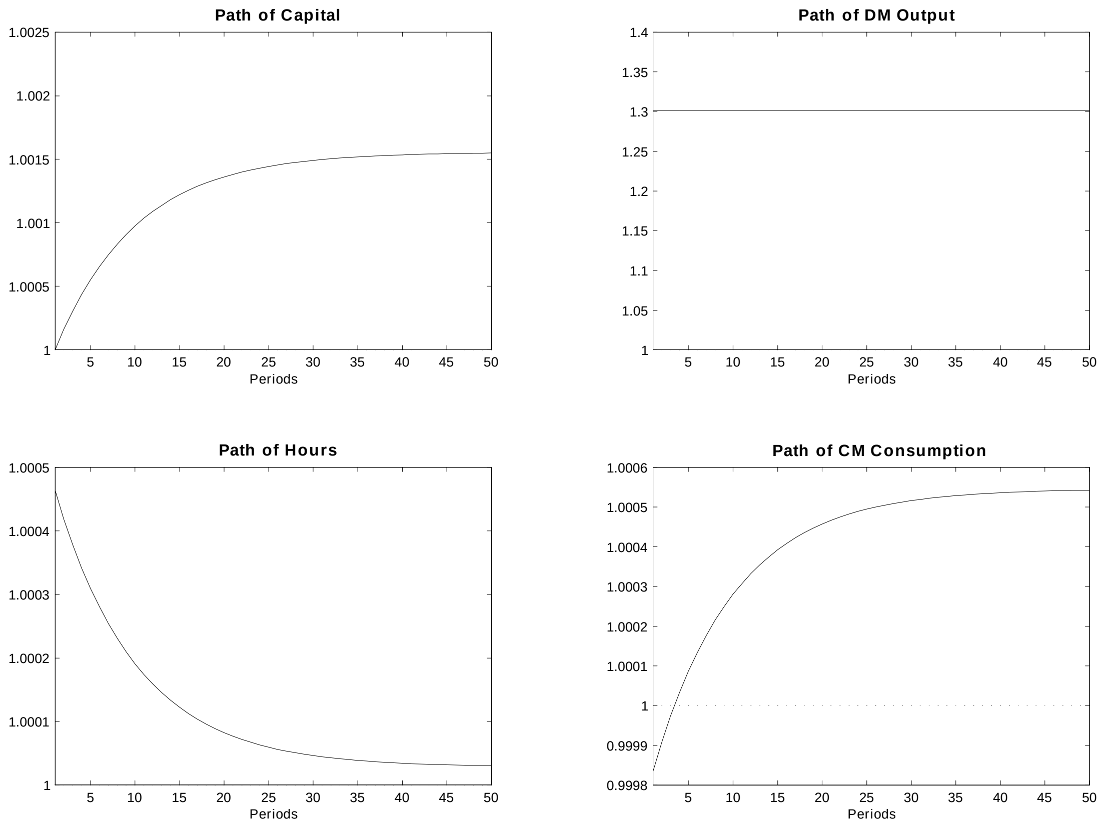
Notes: The dots are US data 1951-2004 and the solid line is the values implied by the steady state of the model, when the interest rate is changed.

Figure 3 - Decision Rules and Value Function: Bargaining Version



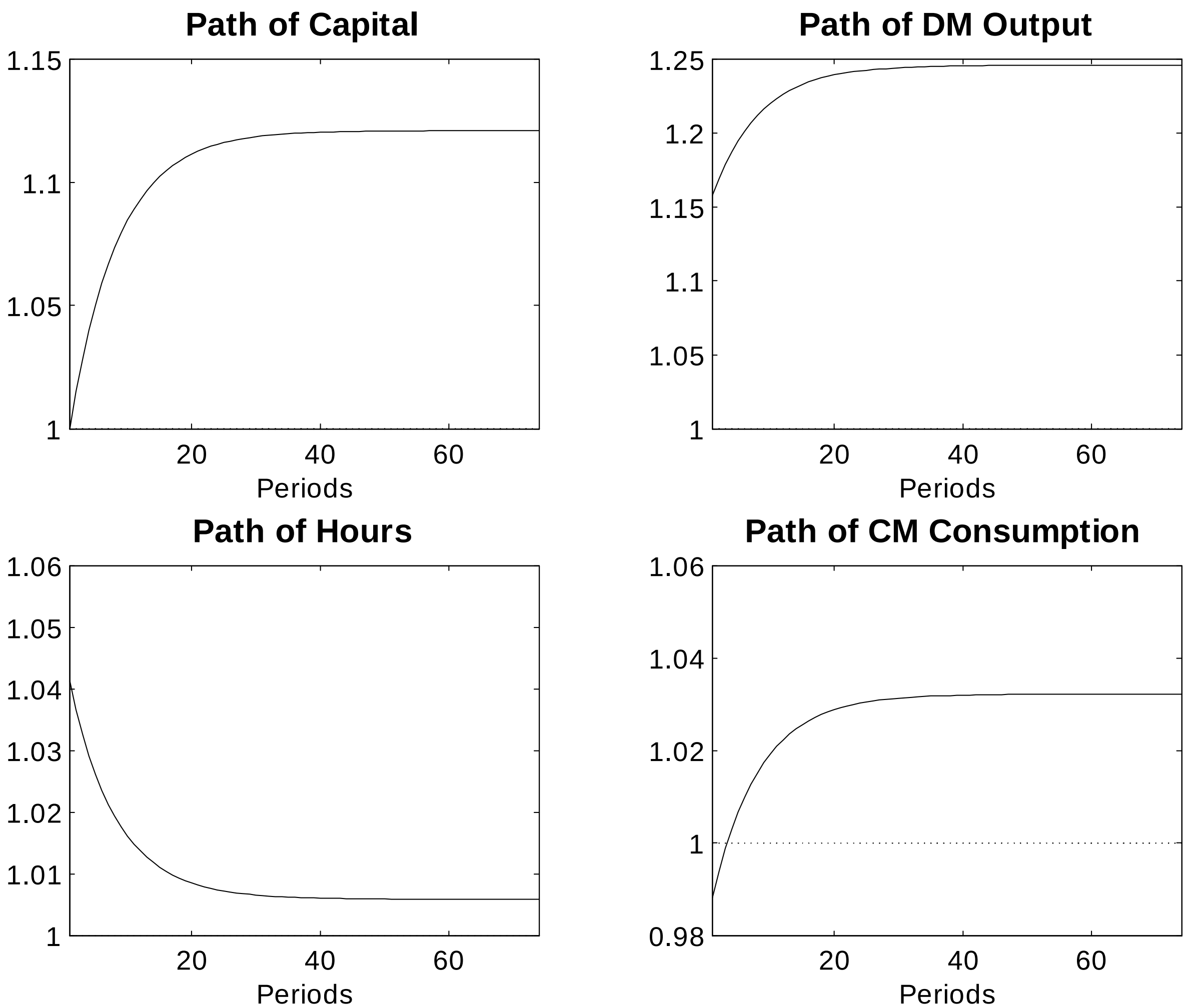
Notes: The x-axis cover $\pm 15\%$ of the steady state of capital. The thin horizontal lines show the steady state values of each variable. The dashed line in the plot of $k'(k)$ is the 45 degree line.

Figure 4 - Transition Path after Reduction in Inflation from 10% to Friedman Rule - Bargaining Version



Notes: Unity on the y-axis refer to the initial steady state.

Figure 5 - Transition Path after Reduction in Inflation from 10% to Friedman Rule - Competitive Pricing Version



Notes: Unity on the y-axis refer to the initial steady state.