Design Limits and Dynamic Policy Analysis

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February 13, 2006

JEL Classification Codes: C52, E6

Keywords: design limits, stabilization policy, robustness, model uncertainty

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Abstract

This paper characterizes the frequency domain properties of linear systems with feedback control rules. The goal of the analysis is to derive restrictions on how feedback rules restrict the frequency by frequency fluctuations that underlie a time series of state variables. The tradeoffs are known in the control literature as design limits. We extend existing results in the control theory literature to account for discrete time bivariate systems with rational expectations. Our basic methods provide ways to understand how fluctuations at different frequencies are subject to tradeoffs via the choice of a feedback rule. Applications are presented to the analysis of robust policy design and frequency-specific Phillips curves.

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I. Introduction

This paper explores a set of constraints on the effects of control policies on fluctuations from the perspective of the frequency domain. Aspects of these constraints were initially discussed in Brock and Durlauf (2004,2005) but otherwise do not appear to have been previously explored in economics contexts. The constraints we study represent fundamental limits on the effects of alternative policies in the sense that they describe how frequency-specific tradeoffs in volatility apply to all linear feedback rules.

The sorts of constraints we explore may be illustrated in the following example. Suppose one is considering how different controls affect the variance of a state variable x_t. Underlying the statistic $\text{var}\left(x_t \middle| C\right)$, the variance of the process given a control, is the spectral density of x given the rule, $f_{x|C}\left(\omega\right)$, as of course the variance is derivative of the spectral density, since

$$\text{var}\left(x_t \middle| C\right) = \int_{-\pi}^{\pi} f_{x|C}\left(\omega\right) d\omega . \tag{1}$$

In fact, the spectral representation of the variance of the state means one can understand the sum of the variances from random and orthogonal sine and cosines of different frequencies. By implication, calculations of the effects of a rule on the overall variance mask the effects on fluctuations at the different frequencies in $\left[-\pi,\pi\right]$. Further, eq. (1) hints at the idea that a rule that minimizes the overall variance may exacerbate fluctuations at certain frequencies. A major goal of this paper is to determine under what circumstances this must happen and what forms such fundamental tradeoffs take. In the control literature, these tradeoffs are known as *design limits*.

Design limits are a well established area of study in control theory.[1] Results of this type are sometimes known as Bode integral constraints, after Hendrik Bode who first proposed them in the 1930's. The great bulk of the work in control theory focuses on single-input, single-output (SISO) systems. One methodological contribution of this

[1]Our description of linear systems owes much to the formulation in Kwakernaak and Sivan (1972), especially chapter 6.

paper is that we derive frequency tradeoffs for multiple-input multiple-output (MIMO) systems. While there does exist a set of disparate results in the control literature on frequency tradeoffs for multivariate systems, this work has largely been done for continuous time systems.[2] One methodological contribution of this paper is that we derive frequency tradeoffs for discrete time. A second methodological contribution is that we study these tradeoffs when expectations of future state variables affect current values; a property that, while of course natural for economic models, does not arise in engineering contexts. Our focus will be on two-input two-output MIMO systems. We defer consideration of arbitrary dimensional systems to future work, noting here that the 2×2 case captures a range of important contexts, most notably the evaluation of macroeconomic stabilization policy, which will be the focus of our application of the general methods.

Why are frequency specific tradeoffs of interest to a policymaker? One reaction to the proposal that policymakers face frequency-by-frequency constraints might be that these constraints are irrelevant if the objective of a policymaker is to minimize the overall variance of some combination of states and controls of the system; such loss functions are standard in the literature on evaluating monetary policy rules. We argue that our results are of interest for several reasons. First, there is no principled reason why policymaker loss functions should only depend on the overall variances of variables of interest, and in fact nonseparable preferences for policymakers can lead to difference losses for different frequency-specific fluctuations. Examples of this property are found in Otrok (2001) and Otrok, Ravikumar, and Whiteman (2002), Second, there are classes of problems for which the frequency restrictions matter, even if loss functions only depend on unconditional variances. Specifically, evaluating the robustness of policy rules in the face of model uncertainty may be facilitated using the constraints we describe; an initial example of such an analysis is Brock and Durlauf (2005).

The use of frequency domain methods is not original per se. One classic example is Hansen and Sargent (1980)'s use of *z*-transform methods to translate time domain expectations into the frequency domain and thereby solve for testable restrictions of rational expectations models. Another important contribution is Whiteman's (1985,1986)

[2]See Skogestad and Postlethwaite (1996) for a survey.

work on spectral utility and the frequency domain analysis of the effects of policies; this work is closest in spirit to ours, although it does not address the issue of frequency-specific tradeoffs. More recently, frequency methods are proven important in the development of robustness, cf. Sargent (1999). Yet another interesting application is developed in Kasa (2000) and Kasa, Walker and Whiteman (2004) which show how frequency domain methods may be used respectively to show how to characterize how individuals form beliefs about the beliefs of others and the equilibrium characterization of prices in markets with asymmetric beliefs. That being said, frequency domain approaches continue to be far less popular than time domain methods for analyzing macroeconomic dynamics. We believe the methods developed here complement these other papers in demonstrating that frequency domain approaches have an important role in understanding stabilization policy. While, in principle, one can always translate results from the frequency domain to the time domain and vice versa, the results we exploit are an example in which working in the frequency domain is relatively straightforward whereas it would appear that the same analysis in the time domain may well be intractable.[3]

Section 2 provides an analysis of three classes of models: backwards-looking MIMO systems, forwards-looking MIMO systems, and SIMO (single input, multiple output) systems. We also consider tradeoffs between frequency-specific variances of states and controls. Section 3 provides applications of these methods. First, we consider the derivation of robust feedback rules. Second, we develop some frequency-specific Phillips curves. Section 4 provides summary and conclusions. Appendices follow which contains proofs of various claims made in the text as well as some examples of general findings.

2. Design limits in multivariate systems

[3]For example, the Bode integral constraint, which we exploit in the subsequent analysis, has an extremely convoluted time domain representation for a SISO system, cf. Iglesias (2001) equation 3.2 and surrounding discussion.

i. backwards looking models

We first consider a backward looking system, i.e. one where expectations do not directly enter into the law of motion for the states. Letting, x_t denote a 2×1 vector of states, u_t a 2×1 vector of controls, and ε_t a 2×1 vector of innovations that is weakly stationary across time, the canonical law of motion for a backwards looking system is

$$A_0 x_t = A(L)x_{t-1} + B(L)u_t + \varepsilon_t . \tag{2}$$

In general, the matrix A_0 will possess off diagonal elements because of contemporary interdependences between the states; without loss of generality, we write the matrix as

$$A_0 = \begin{pmatrix} 1 & a_{0,2} \\ a_{0,3} & 1 \end{pmatrix}.$$

The moving averaging representation of ε_t is

$$\varepsilon_t = W(L)w_t . \tag{3}$$

We assume that each element of $W(L)$ may be written as the ratio of two finite dimensional polynomials,[4] i.e.

$$W(L) = \begin{pmatrix} \frac{w_{n,1}(L)}{w_{d,1}(L)} & \frac{w_{n,2}(L)}{w_{d,2}(L)} \\ \frac{w_{n,3}(L)}{w_{d,3}(L)} & \frac{w_{n,4}(L)}{w_{d,4}(L)} \end{pmatrix}. \tag{4}$$

[4]This assumption means that ε_t possesses a rational spectral density matrix. See Hansen and Sargent (1983) for an example of how rational spectral densities have been used in econometrics.

Notice that we do not require the moving average representation to be fundamental. The reason for this is that our interpretation of the backwards looking model is that it is a structural description of a system.[5]

Our analysis focuses on linear feedback rules of the form

$$u_t = U(L) x_{t-1}. \tag{5}$$

Each choice of this polynomial will produce a different spectral density matrix for the state variable, via the law of motion

$$A_0 x_t = A(L) x_{t-1} + B(L) U(L) x_{t-1} + \varepsilon_t. \tag{6}$$

The moving average representation of the system is

$$x_t = \left(A_0 - \left(A(L) + B(L)U(L)\right)L\right)^{-1} W(L) w_t. \tag{7}$$

This implies that the spectral density matrix for x_t, given a choice of $U(L)$, is

$$f_{x|C}(\omega) = D^C\left(e^{-i\omega}\right)\Sigma_w D^C\left(e^{-i\omega}\right)', \tag{8}$$

where Σ_w is the variance covariance matrix of *w* and

$$D^C\left(e^{-i\omega}\right) = \left(A_0 - \left(A\left(e^{-i\omega}\right) + B\left(e^{-i\omega}\right)U\left(e^{-i\omega}\right)\right)e^{-i\omega}\right)^{-1} W\left(e^{-i\omega}\right). \tag{9}$$

[5]See Fernandez-Villaverde, Rubio-Ramirez, and Sargent (2005) for a comprehensive analysis of the relationship between unrestricted vector autoregressions and structural models, in which invertibility of analogs to $W(L)$ plays a key role.

The superscript C is used because of the dependence of the moving average representation on the choice of control. Each choice of the polynomial $U(L)$ will produce a different spectral density matrix for the state variable vector. Our objective is to describe these limits.

One way to understand the effects of a control role is via the way that $U\left(e^{-i\omega}\right)$ maps to $D^C\left(e^{-i\omega}\right)$. Put differently, the feasible outcomes a policymaker can produce via alternative control rules may be interpreted as the support of the mapping of the set of possible feedback rules to $D^C\left(e^{-i\omega}\right)$. Underlying our calculations of this type is a lemma due to Wu and Jonckhere (1988) which describes the properties of the integrals of logarithms of simple squared polynomials. We provide a slightly different statement of the theorem and a proof in the Appendix.

Lemma 1. (Wu and Jonkheere)

$$\int_{-\pi}^{\pi} \log\left|e^{i\omega}-r\right|^2 d\omega = 0 \text{ if } |r|<1,\ 2\pi\log|r|^2 \text{ otherwise}. \tag{10}$$

The reason that this lemma is so important in the determination of fundamental limits to policies is that in linear environments, the various objects of interest, which amount to products of various polynomials with their associated inner products, can be reduced to products of inner products of simple polynomials; the restrictions on the values of the polynomials will turn out to be described by (10).

In the case of restrictions on the moving average polynomial $D^C\left(e^{-i\omega}\right)$, we will need to focus on the properties of $W(L)$, specifically

$$\det\left(W(z)\right) = \bar{w}\frac{\prod_{i=1}^{w_{MA}}\left(1-w_t z\right)}{\prod_{i=1}^{w_{AR}}\left(1-\rho_t z\right)}. \tag{11}$$

where w_{MA} is the degree of the polynomial $w_{n,1}(L)w_{n,4}(L)w_{d,2}(L)w_{d,3}(L)-w_{n,2}(L)w_{n,3}(L)w_{d,1}(L)w_{d,4}(L)$, w_{AR} is the degree of the polynomial $w_{d,1}(L)w_{d,4}(L)w_{d,2}(L)w_{d,3}(L)$ and $\overline{w}$ is the ratio of the zero degree coefficients on the two polynomials above. Since the innovations ε_t are second order stationary, the roots ρ_i all lie inside the unit circle. However, the roots w_i may lie outside the unit circle as we have not assumed the shocks are fundamental.

Our first result characterizes the feasible values of $D^C\left(e^{-i\omega}\right)$.

Theorem 1. Design limits on the MA polynomial in a backwards looking MIMO model

For the system described by eq. (2), the Fourier transform of the associated sensitivity matrix moving average coefficients, $D^C\left(e^{-i\omega}\right)$, must fulfill

$$\int_{-\pi}^{\pi}\log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right)d\omega = K_w, \tag{12}$$

where

$$K_w = 4\pi\left(\log\overline{w}-\log|a|+\sum_{u_i}\log\left|w_{u_i}\right|\right),\ i\in\{u_i\}\text{ if }|w_i|>1,\ a=\det\left(A_0\right). \tag{13}$$

Pf. See Appendix

The restrictions on possible moving average representations for a controlled system may be elucidated by comparing the properties of the law of motion for the state vector when a control is present with the law of motion when there is no control, i.e. $u_t = 0\ \ \forall t$. The uncontrolled system is simply

$$A_0 x_t = A(L)\,x_{t-1} + \varepsilon_t\,. \tag{14}$$

One can understand the effects of the choice of a control via the differences between the spectral density of the controlled and uncontrolled process, i.e. $f_{x|C}(\omega)$ versus $f_{x|NC}(\omega)$. To make this comparison, first define the spectral density of the no-control system as,

$$f_{x|NC}(\omega) = D^{NC}\left(e^{-i\omega}\right)\Sigma_w D^{NC}\left(e^{-i\omega}\right)', \tag{15}$$

where

$$D^{NC}\left(e^{-i\omega}\right) = \left(A_0 - A\left(e^{-i\omega}\right)e^{-i\omega}\right)^{-1} W\left(e^{-i\omega}\right). \tag{16}$$

Following the control literature, one defines a sensitivity matrix $S\left(e^{-i\omega}\right)$ via the way in which the control transforms $D^{NC}\left(e^{-i\omega}\right)$ into $D^{C}\left(e^{-i\omega}\right)$, i.e.

$$S\left(e^{-i\omega}\right) = D^{C}\left(e^{-i\omega}\right)D^{NC}\left(e^{-i\omega}\right)^{-1}. \tag{17}$$

This allows one to characterize the effects on the spectral density of a given control by

$$f_{x|C}(\omega) = S\left(e^{-i\omega}\right) f_{x|NC}(\omega)\, S\left(e^{-i\omega}\right)'. \tag{18}$$

This formulation makes clear why in the control literature, the sensitivity function is said to shape the state vector.

Each $D^{C}\left(e^{-i\omega}\right)$ maps into a $S\left(e^{-i\omega}\right)$, hence for the policymaker one can think of the choice of control as the choice of a sensitivity function and any constraints on $D^{C}\left(e^{-i\omega}\right)$ in turn may be translated into constraints on $S\left(e^{-i\omega}\right)$. To understand the constraints on the sensitivity function, notice that,

$$\int_{-\pi}^{\pi} \log\left(\left|\det S\left(e^{-i\omega}\right)\right|^2\right) d\omega =$$
$$\int_{-\pi}^{\pi} \log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right) d\omega - \int_{-\pi}^{\pi} \log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right) d\omega. \quad (19)$$

The first term is explicitly characterized by Theorem 1. The second term is implicitly characterized by Theorem 1 and, defining λ_i as the eigenvalues of the characteristic polynomial of the uncontrolled system, equals

$$\int_{-\pi}^{\pi} \log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_w - 4\pi \sum_{v_i} \log\left|\lambda_{v_i}\right| \; i \in \{v_i\} \text{ if } \left|\lambda_i\right| > 1. \quad (20)$$

Combining (13) and (20) immediately provides Theorem 2.

Theorem 2. Design limits on the sensitivity matrix for a backwards looking MIMO model

For the system described by eq. (2), the associated sensitivity matrix $S\left(e^{-i\omega}\right)$ must fulfill

$$\int_{-\pi}^{\pi} \log\left(\left|\det S\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_B, \quad (21)$$

where

$$K_B = 4\pi \sum_{v_i} \log\left|\lambda_{v_i}\right| \; i \in \{v_i\} \text{ if } \left|\lambda_i\right| > 1. \quad (22)$$

This expression has several properties of interest.

First, $K_B = 0$ whenever the unconstrained system is stable. This means that for a large class of models, the constraint on the sensitivity function is identical. More generally, different models may be sorted into equivalence classes with respect to K_B as its value is entirely determined by the unstable roots in the $A(L)$ polynomial. Notice as well that the value of the constraint does not depend on the control rule nor does it depend on $W(L)$, i.e. the (second-order) time series structure of ε_t.

Second, taken together, the facts that a nonzero constraint only occur when the uncontrolled system is unstable and that the magnitudes and number of the unstable roots determine the value of the constraint, indicate that the use of a control to eliminate unit or explosive roots in a system does have a cost in terms of the ability of the policymaker to stabilize fluctuations after these roots have been eliminated. In this sense, trends and cycles do not represent independent aspects of stabilization policy.

Third, policymakers inevitably must trade off variance at different frequencies. Since $|\lambda_i| \geq 1$, it is immediate from (22) that $K_B \geq 0$. This implies that it is impossible for $\left|\det S\left(e^{-i\omega}\right)\right|^2 < 1 \ \forall \omega \in [-\pi, \pi]$ and therefore it is impossible to reduce the variance contributions at all frequencies when one moves from the uncontrolled system to a controlled one. Further, the integral constraint implies that $\left|\det S\left(e^{-i\omega}\right)\right|^2 > 1$ for some interval of frequencies if $\left|\det S\left(e^{-i\omega}\right)\right|^2 < 1$ for another. In order to reduce the variance contributions of one interval of frequencies, it is necessary to increase the variance contributions of some other interval. This tradeoff is fundamental as it cannot be avoided by the choice of control. By implication, minimizing a linear combination of the variances of the elements of x_t will involve trading off frequency specific variance contributions. In other words, variance minimization implies that, even though overall variance is reduced when one integrates across frequencies, for some frequencies, a control that is optimal in this sense leads to greater variance.

ii. forward-looking systems

The law of motion (2) does not embody any direct role for expectations. We next consider systems of the form

$$A_0 x_t = \beta E_t x_{t+1} + A(L) x_{t-1} + B(L) u_t + \varepsilon_t \, . \tag{23}$$

This system is identical to (2) except for the addition of the expectational term $\beta E_t x_{t+1}$. Expectations are assumed to be rational. This means that the state variables will obey an equilibrium moving average representation of the form

$$x_t = F^C(L) w_t = \begin{pmatrix} f_1(L) & f_2(L) \\ f_3(L) & f_4(L) \end{pmatrix} w_t \, , \tag{24}$$

where w_t are fundamental innovations.[6] It is convenient to work with innovations that are contemporaneously uncorrelated. Let $v_t = V w_t$ denote any orthogonalization of the fundamental errors. Then,

$$x_t = F^C(L) V^{-1} v_t = G^C(L) v_t = \begin{pmatrix} g_1(L) & g_2(L) \\ g_3(L) & g_4(L) \end{pmatrix} v_t \, . \tag{25}$$

We note that none of our results depend on the choice of orthogonalization.

As is well known, forward looking systems can exhibit multiple solutions. To ensure uniqueness of the solution it is necessary to restrict the characteristic polynomial $\det\left(A_0 z - \beta - z^2 \left(A(z) + B(z) F(z)\right)\right)$. The following lemma formalizes this; while the result is closely related to Whiteman (1983), the particular claim is new relative to that work.

Lemma 2. Uniqueness of forward-looking solution

The system (23) has a unique square summable moving average solution in the space of v_t's when *exactly two roots* of the characteristic polynomial

[6]A proof of uniqueness appears in the technical appendix. While related to Whiteman (1983), the exact result appears to be new.

$$\det\left(A_0 z-\beta-z^2\left(A(z)+B(z)F(z)\right)\right) \tag{26}$$

are inside the unit circle. If more than two roots are inside the unit circle a square summable moving average solution in the space of v_t's does not exist. If less than two roots are inside the unit circle there exist multiple solutions.

Pf. See appendix.

The rational expectations assumption of course places structure on the individual $g_i(L)$ elements. For our purposes, what matters is that each $g_i(L)$ may be written as a ratio of finite polynomials with common denominator up to $W(L)$; see the technical appendix for a proof that (25) may be written as

$$G^C(z)=\frac{1}{g_d^C(z)}\begin{pmatrix} g_{n,1}^C(z) & g_{n,2}^C(z) \\ g_{n,3}^C(z) & g_{n,4}^C(z) \end{pmatrix}. \tag{27}$$

Here, the subscripts n and d refer to numerator and denominator. The denominator polynomial $g_d^C(L)$ is the characteristic polynomial of the system; define g_d as its zero degree coefficient for later use. Given the assumption that policies are of the feedback form, this coefficient cannot be influenced by the policy choice. Similarly, we define g_n as the coefficient on the zero degree of the polynomial $g_{n,1}^C(L)g_{n,4}^C(L)-g_{n,2}^C(L)g_{n,3}^C(L)$. The form (27) together with the above definitions is useful because it allows us to prove

Theorem 3. Design limits on the MA polynomial in a forwards-looking MIMO model

The moving average coefficients of a controlled system (23) must obey

$$\int_{-\pi}^{\pi} \log\left(\left|\det G^C\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_{w,F}\,, \tag{28}$$

where

$$K_{w,F} = 4\pi\left(\log\left|g_n^C\right| - 2\log\left|g_d\right| + \sum_{u_i} \log\left|g_{n,u_i}^C\right| \right),\ i \in \{u_i\} \text{ if } \left|g_{n,i}\right| > 1\,. \tag{29}$$

In identifying restrictions on the sensitivity function for this system, we once again define a system with no control, i.e.

$$A_0 x_t = \beta E_t x_{t+1} + A(L) x_{t-1} + \varepsilon_t \tag{30}$$

and model the associated law of motion as

$$x_t = G^{NC}(L) v_t = \frac{1}{g_d^{NC}(z)} \begin{pmatrix} g_{n,1}^{NC}(z) & g_{n,2}^{NC}(z) \\ g_{n,3}^{NC}(z) & g_{n,4}^{NC}(z) \end{pmatrix} v_t\,. \tag{31}$$

The sensitivity function is defined by

$$S\left(e^{-i\omega}\right) = G^C\left(e^{-i\omega}\right) G^{NC}\left(e^{-i\omega}\right)^{-1} \tag{32}$$

so its determinant may be written

$$\det S\left(e^{-i\omega}\right)=\det G^{C}\left(e^{-i\omega}\right)\det G^{NC}\left(e^{-i\omega}\right)^{-1}=$$

$$\left(\frac{g_{n,1}^{C}\left(e^{-i\omega}\right)g_{n,4}^{C}\left(e^{-i\omega}\right)-g_{n,2}^{C}\left(e^{-i\omega}\right)g_{n,3}^{C}\left(e^{-i\omega}\right)}{g_{d}^{C}\left(e^{-i\omega}\right)^{2}}\right)\left(\frac{g_{d}^{NC}\left(e^{-i\omega}\right)^{2}}{g_{n,1}^{NC}\left(e^{-i\omega}\right)g_{n,4}^{NC}\left(e^{-i\omega}\right)-g_{n,2}^{NC}\left(e^{-i\omega}\right)g_{n,3}^{NC}\left(e^{-i\omega}\right)}\right)=$$

$$\left(\frac{g_{d}^{NC}\left(e^{-i\omega}\right)^{2}}{g_{d}^{C}\left(e^{-i\omega}\right)^{2}}\right)\left(\frac{g_{n,1}^{C}\left(e^{-i\omega}\right)g_{n,4}^{C}\left(e^{-i\omega}\right)-g_{n,2}^{C}\left(e^{-i\omega}\right)g_{n,3}^{C}\left(e^{-i\omega}\right)}{g_{n,1}^{NC}\left(e^{-i\omega}\right)g_{n,4}^{NC}\left(e^{-i\omega}\right)-g_{n,2}^{NC}\left(e^{-i\omega}\right)g_{n,3}^{NC}\left(e^{-i\omega}\right)}\right)= \tag{33}$$

$$\left(\frac{g_{d}^{2}\left(\Pi_{j=1}^{d^{NC}}\left(1-g_{d,j}^{NC}e^{-i\omega}\right)\right)^{2}}{g_{d}^{2}\left(\Pi_{j=1}^{d^{C}}\left(1-g_{d,j}^{C}e^{-i\omega}\right)\right)^{2}}\right)\left(\frac{g_{n}^{C}\Pi_{j=1}^{n^{C}}\left(1-g_{n,j}^{C}e^{-i\omega}\right)}{g_{n}^{NC}\Pi_{j=1}^{n^{NC}}\left(1-g_{n,j}^{NC}e^{-i\omega}\right)}\right)$$

In the final line of (33), the first ratio also appears in the calculation of the constraints for the sensitivity function of the backwards system as it is a ratio of simple polynomials based on the poles of the controlled and the uncontrolled system. The second ratio incorporates elements of the law of motion that did not affect the sensitivity function for the backwards-looking case. The application of a control can affect the value of g_n^C as well as the location of the zeros $g_{n,j}^C$.

In parallel with Theorem 2, (33) leads immediately to Theorem 4.

Theorem 4. Design limits on the sensitivity function in a forwards-looking MIMO model

The sensitivity function of a controlled system (23) must obey

$$\int_{-\pi}^{\pi}\log\left(\left|\det S\left(e^{-i\omega}\right)\right|^{2}\right)d\omega=K_{F}\,, \tag{34}$$

where

$$K_F = 4\pi\left(\log\left|g_n^C\right| - \log\left|g_n^{NC}\right| + \sum_{v_i}\log\left|g_{d,v_i}^{NC}\right| + \sum_{u_i^C}\log\left|g_{n,u_i}^C\right| - \sum_{u_i^{NC}}\log\left|g_{n,u_i}^{NC}\right| \right), \tag{35}$$
$$i \in \{v_i\} \text{ if } \left|g_{d,i}^{NC}\right| > 1,\; i \in \{u_i^C\} \text{ if } \left|g_{n,i}^C\right| > 1 \text{ and } i \in \{u_i^{NC}\} \text{ if } \left|g_{n,i}^{NC}\right| > 1.$$

From the perspective of design limits, there are several important differences between this case and the backwards looking case.

First, in the presence of an expectational component, the sensitivity function constraint K_F can be negative. This means that it is possible for a control rule to reduce *all* variance contributions relative to an uncontrolled system.

Second, expectations also affect the nature of the constraint value K_F as the terms associated with $\sum_{u_i^C}\log\left|g_{n,u_i}^C\right| - \sum_{u_i^{NC}}\log\left|g_{n,u_i}^{NC}\right|$ do not have an analog in the backwards looking case. These terms are present when $G(L)$ is not invertible, so that the fundamental innovations cannot be recovered from equilibrium realizations of the state variables. Futia (1981) and Hansen and Sargent (1991) refer to this noninvertibility as one where the equilibrium behavior of the state variables is non-revealing; the idea is that the fundamental shocks to a system do not correspond to the structural innovations. This additional constraint may thus be understood as arising from the fact that in a non-revealing equilibrium, policymakers must employ feedback rules that are conditioned on an information set that is coarser than the one that agents employ in making decisions. We provide an example that shows how the application of a control can affect the value of g_n^C as well as the location of the zeros $g_{n,i}^C$ in Appendix II.

iii. SIMO systems

Our analysis has assumed that there are 2 distinct control variables to the policymaker. There are important macroeconomic contexts where this is not the case. For example, a standard question in the analysis of monetary policy rules concerns the effects of different federal funds rate rules on inflation and unemployment, models of this type are standard in Taylor (1999).

We call a 2×2 system MIMO if the policymaker can apply a control for each state variable, we call a system SIMO if only one control variable is available to the policymaker. To clarify this, consider the $AR(1)$ specification of the forward looking system (36) with $A_0 = I$:

$$x_t = \beta E_t x_{t+1} + A x_{t-1} + B u_t + \varepsilon_t \tag{37}$$

where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{38}$$

Under the MIMO assumption the application of a feedback rule $u_t = F x_{t-1}$ is able to change all the elements of the matrix A turning it into A_M^* ,

$$A_M^* = \begin{pmatrix} a_1^* & a_2^* \\ a_3^* & a_4^* \end{pmatrix} \tag{39}$$

Under the SIMO assumption the application of a similar feedback rule can change only the elements of one row of A turning it into A_S^*,

$$A_S^* = \begin{pmatrix} a_1 & a_2 \\ a_3^* & a_4^* \end{pmatrix}. \tag{40}$$

We divide the analysis in two parts: first we consider backward looking systems ($\beta = 0$), then we generalize to forward looking systems ($\beta \neq 0$) (**To be completed**).

In the backward looking $AR(1)$ case, the sensitivity matrix function is:

$$S(L) = \left(I - A^* L\right)^{-1} \left(I - AL\right). \tag{41}$$

Substituting A_M^* and A_S^* in turn it is possible to show that

$$\det S_M\left(L\right)=\frac{\det\left(I-AL\right)}{\det\left(I-A^*L\right)}=\frac{\left(1-a_1L\right)\left(1-a_4L\right)-a_2a_3L^2}{\left(1-a_1^*L\right)\left(1-a_4^*L\right)-a_2^*a_3^*L^2} \tag{42}$$

and

$$\det S_S\left(L\right)=\frac{\det\left(I-AL\right)}{\det\left(I-A^*L\right)}=\frac{\left(1-a_1L\right)\left(1-a_4L\right)-a_2a_3L^2}{\left(1-a_1L\right)\left(1-a_4^*L\right)-a_2a_3^*L^2}. \tag{43}$$

Therefore, as long as we evaluate design limits with respect to $\det\left(S\left(e^{-i\omega}\right)\right)$ there is no difference between a MIMO and a SIMO system in the backwards looking case. One has to go deeper into the structure of the sensitivity function to identify an effect of the number of states exceeding the number of controls.

In order to do this it is necessary to analyze how the sensitivity matrix turns the uncontrolled spectral density matrix into the controlled spectral density matrix. A fundamental difference between univariate and multivariate systems is that the control applied to the latter allocates variances at different frequencies according to the combination of spectra and cross-spectra of the uncontrolled processes. The difference between a MIMO and a SIMO system in terms of design limits may be found at this finer level of analysis.

Suppose that the spectral representation of the variance of the controlled state variables is[7]

$$\begin{pmatrix}\operatorname{var}\left(x_1\right)\\ \operatorname{var}\left(x_2\right)\end{pmatrix}=\begin{pmatrix}\sigma_{\varepsilon_1}^2\int_{-\pi}^{\pi}f_{x_1,\varepsilon_1|C}\left(\omega\right)d\omega+\sigma_{\varepsilon_2}^2\int_{-\pi}^{\pi}f_{x_1,\varepsilon_2|C}\left(\omega\right)d\omega\\ \sigma_{\varepsilon_1}^2\int_{-\pi}^{\pi}f_{x_2,\varepsilon_1|C}\left(\omega\right)d\omega+\sigma_{\varepsilon_2}^2\int_{-\pi}^{\pi}f_{x_2,\varepsilon_2|C}\left(\omega\right)d\omega\end{pmatrix}. \tag{44}$$

[7] Notice that a generalized objective function for the policy maker should be applied to a variance-covariance matrix of the state variables. This is indeed the general approach, taken from the control literature, of Hansen and Sargent (2005).

Employing our definition of the sensitivity matrix it is possible to show (see Appendix) that each spectral component in (44) can be written in terms of the spectral density matrix components of the uncontrolled process as follows:

$$f_{x_1,\varepsilon_1|C}(\omega)=\frac{1}{\left|s_D\left(e^{-i\omega}\right)\right|^2}\begin{pmatrix}\left|s_1\left(e^{-i\omega}\right)\right|^2 f_{x_1,\varepsilon_1|NC}(\omega)+\left|s_2\left(e^{-i\omega}\right)\right|^2 f_{x_2,\varepsilon_1|NC}(\omega)+ \\ s_1\left(e^{-i\omega}\right)s_2\left(e^{i\omega}\right)f_{x_1x_2,\varepsilon_1|NC}(\omega)+s_1\left(e^{i\omega}\right)s_2\left(e^{-i\omega}\right)f_{x_2x_1,\varepsilon_1|NC}(\omega)\end{pmatrix}. \quad (45)$$

This generalizes the univariate case $f_C(\omega)=\left|S\left(e^{-i\omega}\right)\right|^2 f_{NC}(\omega)$. Basically, the application of a feedback policy in a multivariate system shapes the spectral decomposition of the driving process-specific component in each state variable as the weighted sum of three terms. The first term captures the contribution of the uncontrolled analogue of the spectrum considered, the second term captures the contribution of the uncontrolled spectrum of the remaining state variable and the third terms captures the effect of the uncontrolled cross-spectra of the state variables involved. Underlying these three terms there is a common effect of the control represented by the common denominator $\left|s_D\left(e^{-i\omega}\right)\right|^2$ which, by the definition of the sensitivity matrix, corresponds to the characteristic polynomial of the controlled system.

The multivariate version of the Bode constraint tells us that the choice of $s_i\left(e^{-i\omega}\right)$ and $s_D\left(e^{-i\omega}\right)$ are constrained by:

$$K=\int_{-\pi}^{\pi}\ln\left|s_1\left(e^{-i\omega}\right)s_4\left(e^{-i\omega}\right)-s_2\left(e^{-i\omega}\right)s_3\left(e^{-i\omega}\right)\right|^2 d\omega-2\int_{-\pi}^{\pi}\ln\left|s_D\left(e^{-i\omega}\right)\right|^2 d\omega. \quad (46)$$

Given the definition of sensitivity matrix the term $\int_{-\pi}^{\pi}\ln\left|s_D\left(e^{-i\omega}\right)\right|^2 d\omega$ is always equal to zero as it corresponds to the determinant of the matrix solution of the controlled system and the constraint can be rewritten as:

$$K = \int_{-\pi}^{\pi} \ln \left| s_1\left(e^{-i\omega}\right) s_4\left(e^{-i\omega}\right) - s_2\left(e^{-i\omega}\right) s_3\left(e^{-i\omega}\right) \right|^2 d\omega . \tag{47}$$

The structure of each controlled spectra as presented in (45) prompts one to consider tighter constraints of the same nature on terms such as $\left| s_1\left(e^{-i\omega}\right) \right|^2$. Indeed, part of the difference between a MIMO and a SIMO system in terms of design limits is found at this finer level of analysis.

To show this, we exploit the fact that a policymaker whose objective function is a linear combination of the variances of the state variables will lead him to a "white noise equivalent" policy, which means that we assume that the objective of the policy maker is to convert the states to white noise. The specific form of the spectral density matrix for the system (48) and (49) is

$$f_{X|NC}(\omega) = \frac{1}{d(\omega)} \begin{pmatrix} \left|1-a_4e^{-i\omega}\right|^2 \sigma_{\varepsilon_1}^2 + a_2^2\sigma_{\varepsilon_2}^2 & \left(1-a_4e^{-i\omega}\right) a_3 e^{i\omega}\sigma_{\varepsilon_1}^2 + \left(1-a_1e^{i\omega}\right) a_2 e^{-i\omega}\sigma_{\varepsilon_2}^2 \\ \left(1-a_4e^{i\omega}\right) a_3 e^{-i\omega}\sigma_{\varepsilon_1}^2 + \left(1-a_1e^{-i\omega}\right) a_2 e^{i\omega}\sigma_{\varepsilon_2}^2 & a_3^2\sigma_{\varepsilon_1}^2 + \left|1-a_1e^{-i\omega}\right|^2 \sigma_{\varepsilon_2}^2 \end{pmatrix} \tag{50}$$

where

$$d(\omega) = \left| \left(1-a_1e^{-i\omega}\right)\left(1-a_4e^{-i\omega}\right) - a_2a_3e^{-2i\omega} \right|^2 . \tag{51}$$

Assuming a MIMO system, it is possible to obtain a white noise spectral density matrix:

$$f_{X|C}(\omega) = \begin{pmatrix} \sigma_{\varepsilon_1}^2 & 0 \\ 0 & \sigma_{\varepsilon_2}^2 \end{pmatrix}, \tag{52}$$

by setting

$$A_M^* = \begin{pmatrix} a_1^* & a_2^* \\ a_3^* & a_4^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{53}$$

Now consider a SIMO system, in which a white noise spectral density matrix cannot be generally be implemented. Suppose in continuity to the previous example, that the policy maker chooses a control policy so that:

$$A_S^* = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \tag{54}$$

where we assume, in order for the controlled system to be stable, that $|a_1| < 1$. Then, under this policy, the spectral density matrix becomes

$$f_{X|C}(\omega) = \begin{pmatrix} \dfrac{\sigma_{\varepsilon_1}^2 + a_2^2\sigma_{\varepsilon_2}^2}{\left|1 - a_1 e^{-i\omega}\right|^2} & \dfrac{a_2 e^{-i\omega}\sigma_{\varepsilon_2}^2}{\left(1 - a_1 e^{-i\omega}\right)} \\ \dfrac{a_2 e^{i\omega}\sigma_{\varepsilon_2}^2}{\left(1 - a_1 e^{i\omega}\right)} & \sigma_{\varepsilon_2}^2 \end{pmatrix}. \tag{55}$$

A standard class of objective functions for linear systems assumes that the policymaker cares only about the variances of the state variables which corresponds to being interested only in the diagonal elements of the above matrix, so we focus on the diagonal elements for now. If we compare the diagonal elements in (52) and (55) we can see that while the lower-right component is white noise, the upper-left has a spectral density controlled by the parameter a_1. This implies that for some frequencies the variance contribution is reduced in comparison to the MIMO case (modulo the effect of the term $a_2^2\sigma_{\varepsilon_2}^2$) and for other frequencies the contribution is increased. However, overall the variance of the state variable corresponding to the upper-left component is higher, in fact, for $|a_1| < 1$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_{\varepsilon_1}^2 + a_2^2\sigma_{\varepsilon_2}^2}{\left|1 - a_1 e^{-i\omega}\right|^2} d\omega = \frac{\sigma_{\varepsilon_1}^2 + a_2^2\sigma_{\varepsilon_2}^2}{\left(1 - a_1^2\right)} > \sigma_{\varepsilon_1}^2. \tag{56}$$

Suppose now that, with the intent of driving the upper-left component towards a white noise spectrum, the policy maker selects $a_3^* \neq 0$. This choice immediately implies that the lower-right component is no longer white noise and that a positive contribution is added at all frequencies in reason of $a_3^2 \sigma_{\varepsilon_1}^2$.

As already remarked, this particular limitation of a SIMO system is not detected by the general type of Bode constraints that we presented for backwards systems. This is a consequence of the strong model independence of the Bode constraint metric in backwards models, or, in other words, of the particular coarseness of such a metric. Let the sensitivity matrix be defined as in (57) then we have (see Appendix):

$$\left|s_1\left(e^{-i\omega}\right)\right|^2 = \left|\left(1-a_1e^{-i\omega}\right)\left(1-a_4^*e^{-i\omega}\right)-a_2^*a_3e^{-2i\omega}\right|^2 \tag{58}$$

$$\left|s_2\left(e^{-i\omega}\right)\right|^2 = \left|\left(1-a_4e^{-i\omega}\right)a_2^*e^{-i\omega}-\left(1-a_4^*e^{-i\omega}\right)a_2e^{-i\omega}\right|^2 \tag{59}$$

$$\left|s_3\left(e^{-i\omega}\right)\right|^2 = \left|\left(1-a_1e^{-i\omega}\right)a_3^*e^{-i\omega}-\left(1-a_1^*e^{-i\omega}\right)a_3e^{-i\omega}\right|^2 \tag{60}$$

$$\left|s_4\left(e^{-i\omega}\right)\right|^2 = \left|\left(1-a_1^*e^{-i\omega}\right)\left(1-a_4e^{-i\omega}\right)-a_2a_3^*e^{-i\omega}\right|^2 \tag{61}$$

Each of these terms couple with the common denominator $s_D\left(e^{-i\omega}\right)$ to shape the spectral allocation for each component of the controlled spectral matrix according to expressions like (45). We evaluate the four terms above according to the MIMO policy (53) and the SIMO policy (54). This gives:

$$\begin{aligned}
&\left|s_{M,1}\left(e^{-i\omega}\right)\right|^2 = \left|1-a_1e^{-i\omega}\right|^2 && \left|s_{S,1}\left(e^{-i\omega}\right)\right|^2 = \left|\left(1-a_1e^{-i\omega}\right)-a_2a_3e^{-2i\omega}\right|^2 \\
&\left|s_{M,2}\left(e^{-i\omega}\right)\right|^2 = a_2^2 && \left|s_{S,2}\left(e^{-i\omega}\right)\right|^2 = a_2^2a_4^2 \\
&\left|s_{M,3}\left(e^{-i\omega}\right)\right|^2 = a_3^2 && \left|s_{S,3}\left(e^{-i\omega}\right)\right|^2 = a_3^2\left|1-a_1e^{-i\omega}\right|^2 \\
&\left|s_{M,4}\left(e^{-i\omega}\right)\right|^2 = \left|1-a_4e^{-i\omega}\right|^2 && \left|s_{S,4}\left(e^{-i\omega}\right)\right|^2 = \left|1-a_1e^{-i\omega}\right|^2\left|1-a_4e^{-i\omega}\right|^2
\end{aligned} \tag{62}$$

If we apply the Bode metric to each of these elements, recalling that $|a_1|<1$ by assumption, then there does exist a difference in the constraint on $\left|s_{S,1}\left(e^{-i\omega}\right)\right|^2$ and $\left|s_{S,2}\left(e^{-i\omega}\right)\right|^2$ with respect to their MIMO analogues. The difference on the latter term is directly related to a_4 while the difference in the former is more subtle and it depends on the location of the zeros of the polynomial

$$1-a_1z-a_2a_3z^2. \tag{63}$$

It is not difficult to find values for a_2 and a_3, even under the assumption that $|a_2a_3|<1$, such that at least one zero of the polynomial (63) is inside the unit circle, which results in an eigenvalues outside the unit circle and thus a Bode metric different from zero. Under these circumstances one has:

$$\int_{-\pi}^{\pi}\ln\left(\left|s_{S,1}\left(e^{-i\omega}\right)\right|^2\right)d\omega > \int_{-\pi}^{\pi}\ln\left(\left|s_{M,1}\left(e^{-i\omega}\right)\right|^2\right)d\omega = 0 \tag{64}$$

which captures the tighter constraint that a SIMO system entails on the spectral density matrix, exemplified by the difference between (52) and (55). Notice that the difference in sensitivity constraints between a MIMO and a SIMO is driven mostly by the value of the uncontrolled coefficients that are controllable by the policy maker, i.e. a_3 and a_4.

In general, the Bode metric applied element-wise to the sensitivity matrix of a SIMO system delivers constraint values that are greater or equal to the analogue values of a MIMO system. The reason is that the fact that there are fewer controls than states means that there are additional constraints on the individual elements of the matrix $S\left(e^{-i\omega}\right)$ in addition to the overall Bode constraint.

3. Applications

i. robustness

Pioneered by Hansen and Sargent (2001, 2003 and especially 2005), one of the most active areas of contemporaneous macroeconomic research focuses on the question of robustness.[8] This work typically focuses on the question of designing rules that account for model uncertainty. The robustness literature has generally focused on control design in the presence of local model uncertainty, which means that one starts with a given baseline model and considers how to design policies when the true model is close to that model.[9] This recent literature has provided a constructive framework for addressing longstanding worries in macroeconomics about how policy should account for ignorance about economic structure, a concern classically expressed in Friedman's (1951) discussion of long and variable lags in monetary policy effects

Brock and Durlauf (2005) explore robustness in the context of a SISO backwards looking model. In this section, we extend that analysis to forwards looking models. Following their treatment (as well as Sargent (1999)) we focus on uncertainty in the structure of the unobservables ε_t. This is done by starting with a baseline spectral density

[8] Examples include Giannoni (2002), Onatski and Stock (2002), Onatski and Williams (2003), and Tetlow and von zur Muehlen (2001).

[9] Analyses of nonlocal model uncertainty include Levin and Williams (2003) and Brock, Durlauf, and West (2003,2006). These studies employ forms of model averaging, in which a policy's effect is assessed based on a weighted average of the losses associated with specific models; Levin and Williams work with unweighted averages whereas Brock, Durlauf, and West work with posterior model probabilities that reflect the relative empirical support for different models.

for the process, $\overline{f}_\varepsilon(\omega)$, and considering possible spectral densities that are local to it in the sense that each lies in a model space *M* defined by

$$M: \int_{-\pi}^{\pi}\left(f_\varepsilon(\omega)-\overline{f}_\varepsilon(\omega)\right)^2 d\omega \leq \xi^2 . \qquad (65)$$

In the absence of model uncertainty, the policymaker chooses a feedback rule to minimize Ex_t^2. In the presence of uncertainty about $f_\varepsilon(\omega)$, the policymaker chooses a feedback rule under the assumption that the least favorable element in *M* is the actual spectral density. Metaphorically, the policymaker assumes that he is facing an adversarial agent whose objective is to maximize the overall loss Ex_t^2. In this game, the adversarial agent chooses a spectral density for the driving process while the policymaker tries to minimize this expression by applying a control at each frequency. Since we are interested in the resulting Nash equilibrium of this game, we model the two agents as choosing their best strategy conditional on an unspecified strategy of the opponent (we look for best responses). We express these strategies as first order Taylor approximation around a benchmark equilibrium i.e. one in which adversarial agent not present. The main result in Brock and Durlauf (2005) is that, in equilibrium, the spectral density chosen by the adversarial agent magnifies the frequencies that are less down-weighted by the strategy of the policymaker under the benchmark driving process. This is formally described by the equilibrium result:

$$f_\varepsilon^*(\omega)=\overline{f}_\varepsilon(\omega)+\xi\frac{\overline{f}_\varepsilon(\omega)^{-1}}{\left\|\overline{f}_\varepsilon(\omega)^{-1}\right\|}+o(\xi). \qquad (66)$$

The intuition for this finding is straightforward. When ε_t is white noise, then the policymaker cannot offset its presence. Hence, the least favorable spectral density shifts the baseline towards a rectangular shape. This means that those frequencies that are initially large are adjusted less than those that are initially small. Our goal in this section is to identify how forward elements affect this solution.

Consider the univariate hybrid system:

$$x_t = \beta E_t x_{t+1} + A(L)x_{t-1} + B(L)U(L)x_{t-1} + \varepsilon_t \qquad (67)$$

where as before $\varepsilon_t = W(L)w_t$. Under the assumptions of section 2 the unique solution to the above expectational difference equation can be represented as a moving average of the disturbance process ε_t

$$x_t = h^C(L)\varepsilon_t \qquad (68)$$

In addition, denote the equilibrium free dynamics transfer function of the system, i.e. the solution under $U(L) = 0$, by $h^{NC}(e^{-i\omega})$. Absent any robustness considerations, a policymaker chooses a control polynomial $U(L)$ in order to minimize the quadratic loss function

$$E(x_t^2) = \int_{-\pi}^{\pi} \left| h^C(e^{-i\omega}) \right|^2 f_\xi(\omega) d\omega \qquad (69)$$

The moving average representation of the solution of a system with rational expectations has the important property that the structure of the disturbance process affects non-trivially the coefficients of the moving average. It follows that, while in backward looking systems the analogue to the term $\left| h^C(e^{-i\omega}) \right|^2$ for a given policy $U(L)$ does not depend on the spectral density $f_\xi(\omega)$, in a hybrid model this is no longer the case. Nevertheless, even for hybrid systems, it is always possible to isolate a component that depends only on the structure of the controlled system that is invariant to the structure of the disturbance process; formally one can always write:

$$h^C(e^{-i\omega}) = d^C(e^{-i\omega})e^C(e^{-i\omega}) \qquad (70)$$

so that

$$\frac{\partial\left|d^{C}\left(e^{-i\omega}\right)\right|^{2}}{\partial f_{\xi}\left(\omega\right)}=0 \tag{71}$$

This factorization is directly suggested by the moving average representation of the solution (68) which, for a simple $AR(1)$ system with control $u_t = Fx_{t-1}$ is (see Appendix 2):

$$\begin{aligned} h^{C}\left(L\right) &= \frac{\beta h_0 - LW\left(L\right)}{\left(\beta - L - \left(A+BF\right)L^2\right)}W\left(L\right)^{-1} \\ &= \frac{\frac{1}{\lambda_1^C}W\left(\frac{1}{\lambda_1^C}\right) - LW\left(L\right)}{\beta\left(1-\lambda_1^C L\right)\left(1-\lambda_2^C L\right)}W\left(L\right)^{-1} \\ &= \frac{\tilde{W}\left(L;\frac{1}{\lambda_1^C}\right)W\left(L\right)^{-1}}{\beta\left(1-\lambda_2^C L\right)} \end{aligned}$$

where the last step is done by canceling the two factors $\left(1-\lambda_1^C L\right)$ that are present both in the denominator and the numerator by construction. Under this example:

$$d^{C}\left(e^{-i\omega}\right) = \frac{1}{\beta\left(1-\lambda_2^C e^{-i\omega}\right)}$$

$$e^{C}\left(e^{-i\omega}\right) = \tilde{W}\left(L;\frac{1}{\lambda_1^C}\right)W\left(L\right)^{-1}$$

and $f_{\xi}\left(\omega\right)=\left|W\left(e^{-i\omega}\right)\right|^{2}$. Notice that $e^{C}\left(e^{-i\omega}\right)$ is in principle a non trivial function of the function $f_{\xi}\left(\omega\right)$, which means that an expression of the form $\frac{\partial\left|e^{C}\left(e^{-i\omega}\right)\right|^{2}}{\partial f_{\xi}\left(\omega\right)}$ is not a standard derivative but rather a Fréchet derivative (see Appendix for details). The factorization (72) is unique up to a constant which is neutral for our arguments. Similarly, when $U\left(L\right)=0$ one has:

$$h^{NC}\left(e^{-i\omega}\right) = d^{NC}\left(e^{-i\omega}\right) e^{NC}\left(e^{-i\omega}\right) \tag{73}$$

where:

$$\frac{\partial \left| d^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)} = 0 . \tag{74}$$

Equation (75) suggests the definition of a policy function of the form:

$$p(\omega) = \left| \frac{d^{C}\left(e^{-i\omega}\right)}{d^{NC}\left(e^{-i\omega}\right)} \right|^2 \tag{76}$$

which summarizes all the characteristics of the control applied by the policymaker through $U(L)$. The control problem for the policymaker can be equivalently stated as choosing the policy function $p(\omega)$ in order to minimize the objective function

$$E\left(x_t^2\right) = \int_{-\pi}^{\pi} p(\omega) \frac{\left| e^{C}\left(e^{-i\omega}\right)\right|^2}{\left| e^{NC}\left(e^{-i\omega}\right)\right|^2} \left| d^{NC}\left(e^{-i\omega}\right)\right|^2 \left| e^{NC}\left(e^{-i\omega}\right)\right|^2 f_\xi(\omega) d\omega . \tag{77}$$

The purpose of moving to this formulation is that we want the expression inside the integral to factor into a component that is a function of the control C and a part that is not but does depend on the choice of $f_\xi(\omega)$. This allows one to set up a minimization problem that allows a straightforward calculation of the optimal policy given, which provides the strategy of the policymaker in the non-cooperative game with the adversarial agent. By formulating the optimal control problem in this way, the results in the previous section provide precise constraints on the policy function $p(\omega)$. Notice that from (76) and (77) the policy function obeys the following property:

$$\frac{\partial p(\omega)}{\partial f_{\xi}(\omega)} = 0. \tag{78}$$

This partial integral says that the structure of the policy rule does not depend on $f_{\xi}(\omega)$, of course the optimal policy choice will do so. The key role of the presence of the forward looking components in terms of the control policy is that the *effect* of the control leaps into the term $e^{C}\left(e^{-i\omega}\right)$ and it thereby affects the loss function. Let $\zeta(\omega) \equiv \frac{\left|e^{C}\left(e^{-i\omega}\right)\right|^2}{\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}$, the key property of this term is that it depends *directly* on both the action of the policy maker and the action of the adversarial agent. We denote these effects by the following derivatives:

$$\zeta_{f_{\zeta}}(\omega) \equiv \frac{\partial \zeta(\omega)}{\partial f_{\xi}(\omega)} \neq 0 \quad \text{and} \quad \zeta_{p}(\omega) \equiv \frac{\partial \zeta(\omega)}{\partial p(\omega)} \neq 0. \tag{79}$$

Denoting by $p^{*}(\omega)$ the optimal strategy of the policymaker, taken as given by the adversarial agent, the objective function of the latter can now be written as the function $J\left(p^{*}(\omega), f_{\xi}(\omega)\right)$, i.e.

$$J\left(p^{*}(\omega), f_{\xi}(\omega)\right) = \int_{-\pi}^{\pi} p^{*}(\omega)\zeta(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2 \left|e^{NC}\left(e^{-i\omega}\right)\right|^2 f_{\xi}(\omega)\, d\omega. \tag{80}$$

It follows that the marginal effect of a change in the adversarial agent action $J_{f_{\xi}(\omega)}$ is:

$$J_{f_\xi(\omega)} = p^*(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left(\zeta(\omega)\left|e^{NC}\left(e^{-i\omega}\right)\right|^2+\left(\zeta_{f_\zeta}(\omega)\left|e^{NC}\left(e^{-i\omega}\right)\right|^2+\frac{\partial\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)}\zeta(\omega)\right)f_\xi(\omega)\right)$$

. (81)

In presence of only backward looking components $\left|e^{NC}\left(e^{-i\omega}\right)\right|^2=\left|e^{C}\left(e^{-i\omega}\right)\right|^2=1$ at every frequency so that $\zeta(\omega)=1$, $\zeta_{f_\zeta}(\omega)=0$ and $\left|d^{NC}\left(e^{-i\omega}\right)\right|^2=\left|h^{NC}\left(e^{-i\omega}\right)\right|^2$ which results in $J_{f_\xi(\omega)} = p^*(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2$ which is what Brock and Durlauf (2005) obtain.

Consider now the best strategy for the policymaker. Given the way we defined the policy function and by applying the results of Section 2 we have that

$$\int_{-\pi}^{\pi}\ln\left(\frac{\left|d^{C}\left(e^{-i\omega}\right)e^{C}\left(e^{-i\omega}\right)\right|^2}{\left|d^{NC}\left(e^{-i\omega}\right)e^{NC}\left(e^{-i\omega}\right)\right|^2}\right)d\omega=\int_{-\pi}^{\pi}\ln\left(p(\omega)\zeta(\omega)\right)d\omega=K_F$$

We know from our theorems that the Bode constraint for a hybrid model is model-specific, therefore K_F the term will depend on the choice of the policy $p(\omega)$ in a non-trivial way. The problem of the policy maker is then

$$\min_{p(\omega),\lambda}\int_{-\pi}^{\pi}p(\omega)\zeta(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2 f_\xi(\omega)\,d\omega+\lambda\left(K_F-\int_{-\pi}^{\pi}\ln\left(p(\omega)\zeta(\omega)\right)d\omega\right).\text{(82)}$$

The first order necessary conditions imply

$$\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2 f_\xi(\omega)\left(\zeta(\omega)+p(\omega)\zeta_p(\omega)\right)-\frac{\lambda}{p(\omega)}-\lambda\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)=0\ .\text{(83)}$$

At the baseline spectral density the optimal policy satisfies

$$p^*(\omega)\left(\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2\overline{f}_\xi(\omega)\left(\zeta(\omega)+p^*(\omega)\zeta_p(\omega)\right)-\lambda\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)\right)=\lambda$$

. (84)

This expression makes clear that in general an optimal policy for a hybrid model does not necessarily eliminate all the predictability from the state variable. Indeed, the extra-feedback effect represented by $\zeta_p(\omega)\neq 0$ may lead the policymaker to leave some predictability into the process in order to exploit the presence of forward looking agents that rationally anticipate fluctuations.

The expression (85) is an application of an infinite dimension analogue to Brock and Durlauf (2005) equation 5 which can be written as:

$$f_\xi^*(\omega)=\overline{f}_\xi(\omega)+\xi\frac{J_{f_\xi(\omega)}\left(\omega;\overline{f}_\xi(\omega)\right)}{\left(\int_{-\pi}^{\pi}\left(J_{f_\xi(\omega)}\left(\omega;\overline{f}_\xi(\omega)\right)\right)^2 d\omega\right)^{1/2}}+o(\xi). \quad (86)$$

Combining the above equations,

$$J_{f_\xi(\omega)}\left(\omega;\overline{f}_\xi(\omega)\right)=$$

$$\lambda\frac{\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2\left(\zeta(\omega)+\left(\zeta_{f_\zeta}(\omega)+\frac{\partial\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)}\frac{\zeta(\omega)}{\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}\right)\overline{f}_\xi(\omega)\right)}{\left(\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2\overline{f}_\xi(\omega)\left(\zeta(\omega)+p^*(\omega)\zeta_p(\omega)\right)-\lambda\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)\right)}=$$

$$\frac{\lambda}{\overline{f}_\xi(\omega)}\frac{\left(\zeta(\omega)+\left(\zeta_{f_\zeta}(\omega)+\frac{\partial\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)}\frac{\zeta(\omega)}{\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}\right)\overline{f}_\xi(\omega)\right)}{\left(\left(\zeta(\omega)+p^*(\omega)\zeta_p(\omega)\right)-\frac{\lambda}{\overline{f}_\xi(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)\right)}=$$

$$\frac{\lambda}{\overline{f}_\xi(\omega)}\frac{\left(1+\left(\frac{\zeta_{f_\zeta}(\omega)}{\zeta(\omega)}+\frac{\partial\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)}\frac{1}{\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}\right)\overline{f}_\xi(\omega)\right)}{\left(1+p^*(\omega)\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\lambda}{\overline{f}_\xi(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\zeta(\omega)}\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)\right)}= \qquad (87)$$

Even though this equation is not in closed form, it shows that the extent of which the strategy of the adversarial agent is different from the backward looking case depends on the sensitivity of the term that the two players can affect simultaneously, $\zeta(\omega)$, on their respective strategies and on the effect of the policy on the constraint K_F. Letting $\varsigma_a(\omega)\equiv\frac{\zeta_a(\omega)}{\zeta(\omega)}$, $\lambda_{K_F}(\omega)\equiv\frac{\lambda}{\overline{f}_\xi(\omega)\left|d^{NC}\left(e^{-i\omega}\right)\right|^2\zeta(\omega)}\left(\frac{\zeta_p(\omega)}{\zeta(\omega)}-\frac{\partial K_F}{\partial p(\omega)}\right)$ and , $\delta(\omega)\equiv\left(\frac{\partial\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}{\partial f_\xi(\omega)}\frac{1}{\left|e^{NC}\left(e^{-i\omega}\right)\right|^2}\right)$ we conclude that:

$$f_{\xi}^{*}(\omega) = \bar{f}_{\xi}(\omega) + \varepsilon \frac{\bar{f}_{\xi}(\omega)^{-1} \frac{\left(1+\left(\varsigma_{f_{\xi}}(\omega)+\delta(\omega)\right)\bar{f}_{\xi}(\omega)\right)}{\left(1+\varsigma_{p}(\omega)\,p^{*}(\omega)-\lambda_{K_F}(\omega)\right)}}{\left\| \bar{f}_{\xi}(\omega)^{-1} \frac{\left(1+\left(\varsigma_{f_{\xi}}(\omega)+\delta(\omega)\right)\bar{f}_{\xi}(\omega)\right)}{\left(1+\varsigma_{p}(\omega)\,p^{*}(\omega)-\lambda_{K_F}(\omega)\right)} \right\|} + o(\varepsilon) \tag{88}$$

which generalizes, although not in closed form, the main result in Brock and Durlauf (2005).

Relative to Brock and Durlauf's findings, the main intuitive difference is that an adversarial agent will not necessarily shift the forcing variable process, relative to a given baseline, towards white noise. In turn, the changes that are made will depend on the detailed structure of the model in ways that reflect the feedbacks associated with expectations.

ii. frequency-specific Phillips curves

TO BE COMPLETED

4. Summary and conclusions

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Appendix I. Proofs and Derivations

Proof of lemma 1

Elementary calculations show that

$$\left|e^{i\omega}-r\right|^2 = 1-2\left|r\right|\cos\left(\omega-\theta\right)+\left|r\right|^2 \tag{89}$$

where $\theta=\arg\left(r\right)$. Taking logs and integrating both sides of (89),

$$\int_{-\pi}^{\pi}\log\left|e^{i\omega}-r\right|^2 d\omega=\int_{-\pi}^{\pi}\log\left(1-2\left|r\right|\cos\left(\omega-\theta\right)+\left|r\right|^2\right)d\omega. \tag{90}$$

Focusing on the right hand side of (90), it is apparent that the integral is a periodic function with period 2π, so

$$\int_{-\pi}^{\pi}\log\left(1-2\left|r\right|\cos\left(\omega-\theta\right)+\left|r\right|^2\right)d\omega=\int_{-\pi}^{\pi}\log\left(1-2\left|r\right|\cos\omega+\left|r\right|^2\right)d\omega. \tag{91}$$

From Gradeshtyn and Ryzhik (1965),

$$\int_{-\pi}^{\pi}\log\left(a+b\cos\omega\right)d\omega = 2\pi\log\frac{a+\sqrt{a^2+b^2}}{2}, \tag{92}$$

which means that (90) and (91) imply that

$$\int_{-\pi}^{\pi} \log \left| e^{i\omega} - r \right|^2 d\omega = 2\pi \log \frac{1 + |r|^2 + \sqrt{\left(1 + |r|^2\right)^2 - 4|r|^2}}{2}$$
$$= 2\pi \frac{1 + |r|^2 + \left|\left(1 - |r|^2\right)\right|}{2}. \tag{93}$$

Therefore, if $|r| \leq 1$,

$$\int_{-\pi}^{\pi} \log \left| e^{i\omega} - r \right|^2 d\omega = 2\pi \log \frac{1 + |r|^2 + \left(1 - |r|^2\right)}{2} = 0\,. \tag{94}$$

Whereas if $|r| > 1$,

$$\int_{-\pi}^{\pi} \log \left| e^{i\omega} - r \right|^2 d\omega = 2\pi \log \frac{1 + |r|^2 - 1 + |r|^2}{2} = 2\pi \log |r|^2 \tag{95}$$

which is the required result.

Proof of Theorem 1

From the description of the system,

$$\det D\left(e^{-i\omega}\right) = \frac{\det W\left(e^{-i\omega}\right)}{\det\left(A_0 - A\left(e^{-i\omega}\right)e^{-i\omega}\right)} =$$
$$\frac{1}{\det\left(A_0\right)\Pi_{i=1}^{m}\left(1 - \lambda_i e^{-i\omega}\right)} \overline{w} \frac{\Pi_{i=1}^{w_{MA}}\left(1 - w_i e^{-i\omega}\right)}{\Pi_{i=1}^{w_{AR}}\left(1 - \rho_i e^{-i\omega}\right)}. \tag{96}$$

Therefore,

$$\int_{-\pi}^{\pi} \log\left(\left|\det D\left(e^{-i\omega}\right)\right|^2\right) d\omega = \int_{-\pi}^{\pi} \log\left|\frac{1}{\det\left(A_0\right)\Pi_{i=1}^{m}\left(1-\lambda_i e^{-i\omega}\right)} \overline{w} \frac{\Pi_{i=1}^{w_{MA}}\left(1-w_i e^{-i\omega}\right)}{\Pi_{i=1}^{w_{AR}}\left(1-\rho_i e^{-i\omega}\right)}\right|^2 d\omega =$$

$$\int_{-\pi}^{\pi} \log\left(\frac{1}{\det\left(A_0\right)^2 \Pi_{i=1}^{m}\left|e^{i\omega}-\lambda_i\right|^2} \overline{w}^2 \frac{\Pi_{i=1}^{w_{MA}}\left|e^{i\omega}-w_i\right|^2}{\Pi_{i=1}^{w_{AR}}\left|e^{i\omega}-\rho_i\right|^2}\right) d\omega =$$

$$-\sum_{i=1}^{n}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i\right|^2 d\omega - 4\pi\log\det\left(A_0\right) + 4\pi\log\overline{w} + \sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega - \sum_{i=1}^{w_{MA}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\rho_i\right|^2 d\omega \tag{97}$$

From Lemma 1, $\int_{-\pi}^{\pi}\left|e^{i\omega}-r\right|^2 d\omega = 0$ if $|r|<1$. We have assumed that the driving process is second order stationary which means that $|\rho_i|<1$. Hence the last terms in (97) are 0. The terms of interest are $\sum_{i=1}^{n}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i\right|^2 d\omega$ and $\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega$. Concerning the former, the λ_i's are the eigenvalues of the system. When a control is applied to a system it seems desirable to eliminate any unstable eigenvalues, which means that $\lambda_i < 0\ \forall i$. From Lemma 1 this means $\sum_{i=1}^{n}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i\right|^2 d\omega = 0$. However, when we evaluate the expression for uncontrolled systems, unstable eigenvalues cannot be ruled out, which means that

$$\sum_{i=1}^{n}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i\right|^2 d\omega = 4\pi\sum_{v_i}\log\left|\lambda_{v_i}\right| \ i\in\{v_i\} \text{ if } \left|\lambda_1\right|>1$$

Consider now the latter term. $\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega$. As already noticed, we allow for $|w_i|>1$, therefore we have

$$\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega = 4\pi\sum_{u_i}\log\left|w_{u_i}\right| \ i\in\{u_i\} \text{ if } \left|w_i\right|>1$$

which verifies the theorem. Notice that this last term does not change when a control is applied to an uncontrolled system, and in fact, theorem 2 makes clear that the structure of the driving process does not affect the constraint on the sensitivity function in backwards looking models. What do matter for the constraint are the unstable eigenvalues of the uncontrolled system: as the feedback control must be designed in order to stabilize the system, the freedom in allocating the variance at different frequencies is reduced.

Proof of Lemma 2

Since $x_t = G(L)v_t$, and v_t maps 1 to 1 with w_t, the Wiener-Kolmogorov prediction formula implies that

$$E_t x_{t+1} = L^{-1}\left(G(L) - G_0\right)v_t \tag{98}$$

which, given eq. (98), means that

$$G(L)\Big(A_0 - L^{-1}\beta - \big(A(L) - B(L)U(L)\big)L\Big)v_t = \left(-L^{-1}\beta G_0 + V(L)\right)v_t \tag{99}$$

Taking z -transforms,

$$G(z) = \frac{adj\left(zA_0 - \beta - \left(A(z) - B(z)U(z)\right)z^2\right)}{\det\left(zA_0 - \beta - \left(A(z) - B(z)U(z)\right)z^2\right)}\left(-\beta G_0 + zV(z)\right) \tag{100}$$

Recall that the specific forms of β and G_0 are $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ and $\mathrm{G}_0 = \begin{pmatrix} g_{0,1} & g_{0,2} \\ g_{0,3} & g_{0,4} \end{pmatrix}$ respectively. The term $\left(-\beta G_0 + zV(z)\right)$ on the right hand side therefore may be rewritten

$$-\beta G_0 + zV(z) = -\begin{pmatrix} \beta_1 g_{0,1} + \beta_2 g_{0,3} - zv_1(z) & \beta_1 g_{0,2} + \beta_2 g_{0,4} - zv_2(z) \\ \beta_3 g_{0,1} + \beta_4 g_{0,3} - zv_3(z) & \beta_3 g_{0,2} + \beta_4 g_{0,4} - zv_4(z) \end{pmatrix} \tag{101}$$

The solution $G(L)$ is thus determined up to the elements in G_0. Following Whiteman (1983) we use the requirement of analyticity for each term in $G(z)$ as a set of extra conditions for uniqueness. Define

$$-adj\left(zA_0-\beta-\left(A(z)-B(z)U(z)\right)z^2\right)=\begin{pmatrix} J_1(z) & J_2(z) \\ J_3(z) & J_4(z) \end{pmatrix} \tag{102}$$

and

$$J(z)=\det\left(zA_0-\beta-\left(A(z)-B(z)U(z)\right)z^2\right) \tag{103}$$

Note that under our assumptions, each term $J_i(z)$ is a finite degree polynomial. The candidate solution can be then written as:

$$G(z)=\frac{1}{J(z)}\begin{pmatrix} J_1(z) & J_2(z) \\ J_3(z) & J_4(z) \end{pmatrix}\begin{pmatrix} \beta_1 g_{0,1}+\beta_2 g_{0,3}-zv_1(z) & \beta_1 g_{0,2}+\beta_2 g_{0,4}-zv_2(z) \\ \beta_3 g_{0,1}+\beta_4 g_{0,3}-zv_3(z) & \beta_3 g_{0,2}+\beta_4 g_{0,4}-zv_4(z) \end{pmatrix} \tag{104}$$

The conditions for uniqueness corresponds to each numerator term in this matrix to become zero at the unstable poles of the denominator (zeros inside the unit circle) that we denote by $\bar{z}$. Each $\bar{z}$ provides the associated set of equations:

$$\begin{aligned}
&J_1(\bar{z})\left(\beta_1 g_{0,1}+\beta_2 g_{0,3}-\bar{z}v_1(\bar{z})\right)+J_2(\bar{z})\left(\beta_3 g_{0,1}+\beta_4 g_{0,3}-\bar{z}v_3(\bar{z})\right)=0\\
&J_3(\bar{z})\left(\beta_1 g_{0,1}+\beta_2 g_{0,3}-\bar{z}v_1(\bar{z})\right)+J_4(\bar{z})\left(\beta_3 g_{0,1}+\beta_4 g_{0,3}-\bar{z}v_3(\bar{z})\right)=0\\
&J_1(\bar{z})\left(\beta_1 g_{0,2}+\beta_2 g_{0,4}-\bar{z}v_2(\bar{z})\right)+J_2(\bar{z})\left(\beta_3 g_{0,2}+\beta_4 g_{0,4}-\bar{z}v_4(\bar{z})\right)=0\\
&J_3(\bar{z})\left(\beta_1 g_{0,2}+\beta_2 g_{0,4}-\bar{z}v_2(\bar{z})\right)+J_4(\bar{z})\left(\beta_3 g_{0,2}+\beta_4 g_{0,4}-\bar{z}v_4(\bar{z})\right)=0
\end{aligned} \tag{105}$$

These four equations are actually two blocks of two equations (first and third, second and fourth) that can be solved independently for g_{01}, g_{03} and g_{02}, g_{04} respectively. To see this we focus on the first block which can be written as:

$$\begin{pmatrix} J_1(\bar{z})\beta_1 + J_2(\bar{z})\beta_3 & J_1(\bar{z})\beta_2 + J_2(\bar{z})\beta_4 \\ J_3(\bar{z})\beta_1 + J_4(\bar{z})\beta_3 & J_3(\bar{z})\beta_2 + J_4(\bar{z})\beta_4 \end{pmatrix} \begin{pmatrix} g_{0,1} \\ g_{0,3} \end{pmatrix} = \begin{pmatrix} J_1(\bar{z})\bar{z}v_1(\bar{z}) + J_2(\bar{z})\bar{z}v_3(\bar{z}) \\ J_3(\bar{z})\bar{z}v_1(\bar{z}) + J_4(\bar{z})\bar{z}v_3(\bar{z}) \end{pmatrix} \tag{106}$$

from which it is evident that

$$\begin{vmatrix} J_1(\bar{z})\beta_1 + J_2(\bar{z})\beta_3 & J_1(\bar{z})\beta_2 + J_2(\bar{z})\beta_4 \\ J_3(\bar{z})\beta_1 + J_4(\bar{z})\beta_3 & J_3(\bar{z})\beta_2 + J_4(\bar{z})\beta_4 \end{vmatrix} = 0 \tag{107}$$

In order to have a unique solution for this block system it is therefore necessary to have two zeros inside the unit circle. When this holds, the two equations in two unknowns are

$$\begin{gathered} \begin{pmatrix} J_1(\bar{z}_1)\beta_1 + J_2(\bar{z}_1)\beta_3 & J_1(\bar{z}_1)\beta_2 + J_2(\bar{z}_1)\beta_4 \\ J_1(\bar{z}_2)\beta_1 + J_2(\bar{z}_2)\beta_3 & J_1(\bar{z}_2)\beta_2 + J_2(\bar{z}_2)\beta_4 \end{pmatrix} \begin{pmatrix} g_{0,1} \\ g_{0,3} \end{pmatrix} = \\ \begin{pmatrix} J_1(\bar{z}_1)\bar{z}_1 v_1(\bar{z}_1) + J_2(\bar{z}_1)\bar{z}_1 v_3(\bar{z}_1) \\ J_1(\bar{z}_2)\bar{z}_2 v_1(\bar{z}_2) + J_1(\bar{z}_2)\bar{z}_2 v_3(\bar{z}_2) \end{pmatrix} \end{gathered} \tag{108}$$

which identify a unique solution for g_{01} and g_{03}. The solution method for g_{02} and g_{04} is identical.

Derivation of Equation (27)

Let

$$\det\left(zA_0 - \beta - \left(A(z) - B(z)U(z)\right)z^2\right) = \tilde{g}_d(z). \tag{109}$$

We denote the determinant by $\tilde{g}_d(z)$ to stress that the cancellation of unstable roots that allows the uniqueness of the solution has not yet been considered. The solution matrix can thus be written as

$$G(z)=\frac{1}{\tilde{g}_d(z)}\begin{pmatrix} J_1(z) & J_2(z) \\ J_3(z) & J_4(z) \end{pmatrix}\begin{pmatrix} \beta_1 g_{0,1}+\beta_2 g_{0,3}-zv_1(z) & \beta_1 g_{0,2}+\beta_2 g_{0,4}-zv_2(z) \\ \beta_3 g_{0,1}+\beta_4 g_{0,3}-zv_3(z) & \beta_3 g_{0,2}+\beta_4 g_{0,4}-zv_4(z) \end{pmatrix} \quad (110)$$

Each of the four still-undefined constants in the second matrix on the right hand side are then chosen in order to make $G(z)$ analytic inside the unit circle. Once the constants have been chosen it is possible to write each term of $G(L)$ as having a common denominator whose zeros are all outside the unit circle, we denote it by $g_d(z)$; as noticed, this is true conditional on the structure of $V(L)$, a statement which now we clarify.

Recalling that the $W(L)$, has the form (4) it must be the case that for $V(L)$

$$V(L)=\begin{pmatrix} v_1(L) & v_2(L) \\ v_3(L) & v_4(L) \end{pmatrix}=\begin{pmatrix} \frac{v_{n,1}(L)}{v_{d,1}(L)} & \frac{v_{n,2}(L)}{v_{d,2}(L)} \\ \frac{v_{n,3}(L)}{v_{d,3}(L)} & \frac{v_{n,4}(L)}{v_{d,4}(L)} \end{pmatrix}, \quad (111)$$

where the numerator polynomials are defined so that:

$$v_i(L)=\frac{v_i^n(L)}{v_i^e(L)}=\frac{v_i^n(L)}{\prod_{j=1}^{k_i}\left(1-v_{i,j}^d L\right)}. \quad (112)$$

The terms in the second matrix of the solution matrix will be of the form

$$\beta_1 g_{0,1} + \beta_2 g_{0,3} - z\frac{v_{n,1}(L)}{v_{d,1}(L)} = \frac{v_{d,1}(L)\left(\beta_1 g_{0,1} + \beta_2 g_{0,3}\right) - z v_{n,1}(L)}{v_{d,1}(L)} \equiv \frac{\hat{v}_{n,1}(L)}{v_{d,1}(L)}. \qquad (113)$$

It follows that the form of the solution matrix is

$$\begin{aligned}
G(z) &= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} J_1(z) & J_2(z) \\ J_3(z) & J_4(z) \end{pmatrix} \begin{pmatrix} \frac{\hat{v}_{n,1}(z)}{v_{d,1}(z)} & \frac{\hat{v}_{n,2}(z)}{v_{d,2}(z)} \\ \frac{\hat{v}_{n,3}(z)}{v_{d,3}(z)} & \frac{\hat{v}_{n,4}(z)}{v_{d,4}(z)} \end{pmatrix} \\
&= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} J_1(z)\frac{\hat{v}_{n,1}(z)}{v_{d,1}(z)} + J_2(z)\frac{\hat{v}_{n,3}(z)}{v_{d,3}(z)} & J_1(z)\frac{\hat{v}_{n,2}(z)}{v_{d,2}(z)} + J_2(z)\frac{\hat{v}_{n,4}(z)}{v_{d,4}(z)} \\ J_3(z)\frac{\hat{v}_{n,1}(z)}{v_{d,1}(z)} + J_4(z)\frac{\hat{v}_{n,3}(z)}{v_{d,3}(z)} & J_3(z)\frac{\hat{v}_{n,2}(z)}{v_{d,2}(z)} + J_4(z)\frac{\hat{v}_{n,4}(z)}{v_{d,4}(z)} \end{pmatrix} \\
&= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} \frac{\tilde{g}_1^n(z)}{v_{d,1}(z)v_{d,3}(z)} & \frac{\tilde{g}_2^n(z)}{v_{d,2}(z)v_{d,4}(z)} \\ \frac{\tilde{g}_3^n(z)}{v_{d,1}(z)v_{d,3}(z)} & \frac{\tilde{g}_4^n(z)}{v_{d,2}(z)v_{d,4}(z)} \end{pmatrix} \\
&= \frac{1}{g_d(z)} \begin{pmatrix} \frac{g_{n,1}(z)}{v_{d,1}(z)v_{d,3}(z)} & \frac{g_{n,2}(z)}{v_{d,2}(z)v_{d,4}(z)} \\ \frac{g_{n,3}(z)}{v_{d,1}(z)v_{d,3}(z)} & \frac{g_{n,4}(z)}{v_{d,2}(z)v_{d,4}(z)} \end{pmatrix} \qquad (114)
\end{aligned}$$

The first step is just matrix multiplication, the second step defines the numerators of each term as $\tilde{g}_{n,i}(z)$ since the free parameters are not pinned down yet; the last step applies the requirements for the uniqueness of a solution by choosing the free parameters so as to cancel the zeros of the common denominator $\tilde{g}_d(z)$ inside the unit circle with the zeros of each numerator $\tilde{g}_{n,i}(z)$. Since the disturbance process is assumed stationary, each of the terms $v_{d,i}(z)$ have all their zeros outside the unit circle and therefore they do not play any direct active role in the value of the Bode constraint. They of course play a more subtle role as the zeros (possibly inside the unit circle) of the terms $g_{n,i}(z)$ will depend on them too. The last line of the above equation clarifies the statement that the solution $G(z)$, for the purpose of applying the Wu-Jonckheere results, can be written as

$$G(z) = \frac{1}{g_d(z)} \begin{pmatrix} g_{n,1}(z) & g_{n,2}(z) \\ g_{n,3}(z) & g_{n,4}(z) \end{pmatrix} \tag{115}$$

conditional on the autoregressive structure of $V(L)$ summarized by the polynomial $v_{d,1}(z)v_{d,3}(z)v_{d,2}(z)v_{d,4}(z)$. In fact, notice that

$$\begin{gathered} \det\left[\frac{1}{g_d(z)} \begin{pmatrix} \dfrac{g_{n,1}(z)}{v_{d,1}(z)v_{d,3}(z)} & \dfrac{g_{n,2}(z)}{v_{d,2}(z)v_{d,4}(z)} \\ \dfrac{g_{n,3}(z)}{v_{d,1}(z)v_{d,3}(z)} & \dfrac{g_{n,4}(z)}{v_{d,2}(z)v_{d,4}(z)} \end{pmatrix}\right] = \\ \det\left[\frac{1}{g_d(z)} \begin{pmatrix} g_{n,1}(z) & g_{n,2}(z) \\ g_{n,3}(z) & g_{n,4}(z) \end{pmatrix}\right] \frac{1}{v_{d,1}(z)v_{d,3}(z)v_{d,2}(z)v_{d,4}(z)} \end{gathered} \tag{116}$$

Since the denominator terms have been constructed so that

$$v_{d,1}(z)v_{d,3}(z)v_{d,2}(z)v_{d,4}(z) = \prod_{j=1}^{k} \left(1 - v_j L\right) \tag{117}$$

where k is the sum of the degrees of each denominator term and the v_j's are the eigenvalues associated to the zeros of each denominator term, which are assumed all inside the unit circle, thus the contribution of the last term in (117) to the Bode constraint is null. It is the assumption (112) about the representation embedded in $v_{n,i}(z)$ that allows the results of the theorems to be applicable in presence of a generic $V(L)$. In fact, given that assumption, it will be true in general that $\overline{v}_{n,i} \neq 1$ in

$$v_{n,i}(L) = \bar{v}_{n,i} \prod_{j=1}^{k_{n,i}} \left(1 - v_{n,i,j} L\right) \tag{118}$$

SIMO derivation

Suppose that the spectral representation of the variance of the state variables is[10]

$$\begin{pmatrix} v_{x_1} \\ v_{x_2} \end{pmatrix} = \begin{pmatrix} \int_{-\pi}^{\pi} f_{x_1,\varepsilon_1}(\omega) d\omega & \int_{-\pi}^{\pi} f_{x_1,\varepsilon_2}(\omega) d\omega \\ \int_{-\pi}^{\pi} f_{x_2,\varepsilon_1}(\omega) d\omega & \int_{-\pi}^{\pi} f_{x_2,\varepsilon_2}(\omega) d\omega \end{pmatrix} \begin{pmatrix} \sigma_{\varepsilon_1}^2 \\ \sigma_{\varepsilon_2}^2 \end{pmatrix}, \tag{119}$$

where

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} x_{1,\varepsilon_1}\left(e^{-i\omega}\right) & x_{1,\varepsilon_2}\left(e^{-i\omega}\right) \\ x_{2,\varepsilon_1}\left(e^{-i\omega}\right) & x_{2,\varepsilon_2}\left(e^{-i\omega}\right) \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \tag{120}$$

and

$$\begin{pmatrix} f_{x_1,\varepsilon_1}(\omega) & f_{x_1,\varepsilon_2}(\omega) \\ f_{x_2,\varepsilon_1}(\omega) & f_{x_2,\varepsilon_2}(\omega) \end{pmatrix} \equiv \begin{pmatrix} \left|x_{1,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 & \left|x_{1,\varepsilon_2}\left(e^{-i\omega}\right)\right|^2 \\ \left|x_{2,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 & \left|x_{2,\varepsilon_2}\left(e^{-i\omega}\right)\right|^2 \end{pmatrix}. \tag{121}$$

Then the controlled system can be written as:

$$\begin{aligned} \begin{pmatrix} \pi^C{}_t \\ y^C{}_t \end{pmatrix} &= \begin{pmatrix} s_1\left(e^{-i\omega}\right) & s_2\left(e^{-i\omega}\right) \\ s_3\left(e^{-i\omega}\right) & s_4\left(e^{-i\omega}\right) \end{pmatrix} \begin{pmatrix} x_{1,\varepsilon_1}\left(e^{-i\omega}\right) & x_{1,\varepsilon_2}\left(e^{-i\omega}\right) \\ x_{2,\varepsilon_1}\left(e^{-i\omega}\right) & x_{2,\varepsilon_2}\left(e^{-i\omega}\right) \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \\ &= \begin{pmatrix} s_1\left(e^{-i\omega}\right) x_{1,\varepsilon_1}\left(e^{-i\omega}\right) + s_2\left(e^{-i\omega}\right) x_{2,\varepsilon_1}\left(e^{-i\omega}\right) & s_1\left(e^{-i\omega}\right) x_{1,\varepsilon_2}\left(e^{-i\omega}\right) + s_2\left(e^{-i\omega}\right) x_{2,\varepsilon_2}\left(e^{-i\omega}\right) \\ s_3\left(e^{-i\omega}\right) x_{1,\varepsilon_1}\left(e^{-i\omega}\right) + s_4\left(e^{-i\omega}\right) x_{2,\varepsilon_1}\left(e^{-i\omega}\right) & s_3\left(e^{-i\omega}\right) x_{1,\varepsilon_2}\left(e^{-i\omega}\right) + s_4\left(e^{-i\omega}\right) x_{2,\varepsilon_2}\left(e^{-i\omega}\right) \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \end{aligned} \tag{122}$$

[10] Notice that a generalized objective function for the policy maker should be applied to a variance-covariance matrix of the state variables. This is indeed the general approach, taken from the control literature, of Hansen and Sargent (2005).

It is useful to compare (120) with (122) in order to grasp the effect of a feedback policy in a multivariate (bivariate) system. Each term in the matrix transfer function of the controlled system depends on a linear combination of the transfer functions of the uncontrolled system for both state variables relative to the same term of the driving process vector. In other words, the transfer function $x^C_{1,\varepsilon_1}\left(e^{-i\omega}\right)$ depends on $x_{1,\varepsilon_1}\left(e^{-i\omega}\right)$ and $x_{2,\varepsilon_1}\left(e^{-i\omega}\right)$ combined linearly by $s_1\left(e^{-i\omega}\right)$ and $s_2\left(e^{-i\omega}\right)$. The action in the state variable generated by $x^C_{1,\varepsilon_1}\left(e^{-i\omega}\right)$ is thus a combination of the independent actions of both the uncontrolled state variables and their common action. In fact, the controlled matrix of spectra is

$$\begin{pmatrix} f^C_{x_1,\varepsilon_1}(\omega) & f^C_{x_1,\varepsilon_2}(\omega) \\ f^C_{x_2,\varepsilon_1}(\omega) & f^C_{x_2,\varepsilon_2}(\omega) \end{pmatrix}$$
$$\equiv \begin{pmatrix} \left|s_1\left(e^{-i\omega}\right)x_{1,\varepsilon_1}\left(e^{-i\omega}\right)+s_2\left(e^{-i\omega}\right)x_{2,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 & \left|s_1\left(e^{-i\omega}\right)x_{1,\varepsilon_2}\left(e^{-i\omega}\right)+s_2\left(e^{-i\omega}\right)x_{2,\varepsilon_2}\left(e^{-i\omega}\right)\right|^2 \\ \left|s_3\left(e^{-i\omega}\right)x_{1,\varepsilon_1}\left(e^{-i\omega}\right)+s_4\left(e^{-i\omega}\right)x_{2,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 & \left|s_3\left(e^{-i\omega}\right)x_{1,\varepsilon_2}\left(e^{-i\omega}\right)+s_4\left(e^{-i\omega}\right)x_{2,\varepsilon_2}\left(e^{-i\omega}\right)\right|^2 \end{pmatrix} \tag{123}$$

and each spectral component is of the form:

$$f^C_{x_1,\varepsilon_1}(\omega) = \left|s_1\left(e^{-i\omega}\right)\right|^2\left|x_{1,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 + \left|s_2\left(e^{-i\omega}\right)\right|^2\left|x_{2,\varepsilon_1}\left(e^{-i\omega}\right)\right|^2 + s_1\left(e^{-i\omega}\right)s_2\left(e^{i\omega}\right)x_{1,\varepsilon_1}\left(e^{-i\omega}\right)x_{2,\varepsilon_1}\left(e^{i\omega}\right) + s_1\left(e^{i\omega}\right)s_2\left(e^{-i\omega}\right)x_{1,\varepsilon_1}\left(e^{i\omega}\right)x_{2,\varepsilon_1}\left(e^{-i\omega}\right) \tag{124}$$

Following Sargent (1987, eq. (47) page 268) we notice that the cross-spectrum is defined as:

$$f_{x_1x_2,\varepsilon_1}(\omega) = x_{1,\varepsilon_1}\left(e^{-i\omega}\right)x_{2,\varepsilon_1}\left(e^{i\omega}\right) \text{ and } \quad f_{x_2x_1,\varepsilon_1}(\omega) = x_{1,\varepsilon_1}\left(e^{i\omega}\right)x_{2,\varepsilon_1}\left(e^{-i\omega}\right) \tag{125}$$

Thus each spectrum component of the controlled process can be finally written as:

$$f_{x_1,\varepsilon_1}^{C}(\omega)=\left|s_1\left(e^{-i\omega}\right)\right|^2 f_{x_1,\varepsilon_1}(\omega)+\left|s_2\left(e^{-i\omega}\right)\right|^2 f_{x_2,\varepsilon_1}(\omega)+ \\ s_1\left(e^{-i\omega}\right)s_2\left(e^{i\omega}\right) f_{x_1x_2,\varepsilon_1}(\omega)+s_1\left(e^{i\omega}\right)s_2\left(e^{-i\omega}\right) f_{x_2x_1,\varepsilon_1}(\omega) \tag{126}$$

$$f_{x_1,\varepsilon_2}^{C}(\omega)=\left|s_1\left(e^{-i\omega}\right)\right|^2 f_{x_1,\varepsilon_2}(\omega)+\left|s_2\left(e^{-i\omega}\right)\right|^2 f_{x_2,\varepsilon_2}(\omega)+ \\ s_1\left(e^{-i\omega}\right)s_2\left(e^{i\omega}\right) f_{x_1x_2,\varepsilon_2}(\omega)+s_1\left(e^{i\omega}\right)s_2\left(e^{-i\omega}\right) f_{x_2x_1,\varepsilon_2}(\omega) \tag{127}$$

$$f_{x_2,\varepsilon_1}^{C}(\omega)=\left|S_3\left(e^{-i\omega}\right)\right|^2 f_{x_1,\varepsilon_1}(\omega)+\left|S_4\left(e^{-i\omega}\right)\right|^2 f_{x_2,\varepsilon_1}(\omega)+ \\ S_3\left(e^{-i\omega}\right)S_4\left(e^{i\omega}\right) f_{x_1x_2,\varepsilon_1}(\omega)+S_3\left(e^{i\omega}\right)S_4\left(e^{-i\omega}\right) f_{x_2x_1,\varepsilon_1}(\omega) \tag{128}$$

$$f_{x_2,\varepsilon_2}^{C}(\omega)=\left|S_3\left(e^{-i\omega}\right)\right|^2 f_{x_1,\varepsilon_2}(\omega)+\left|S_4\left(e^{-i\omega}\right)\right|^2 f_{x_2,\varepsilon_2}(\omega)+ \\ S_3\left(e^{-i\omega}\right)S_4\left(e^{i\omega}\right) f_{x_1x_2,\varepsilon_2}(\omega)+S_3\left(e^{i\omega}\right)S_4\left(e^{-i\omega}\right) f_{x_2x_1,\varepsilon_2}(\omega) \tag{129}$$

as claimed in the text.

Appendix 2. Feedback Policies and Non-Revealing Equilibria

In Section 2 we argued that for forward looking models, the design limits faced by a policymaker depend on the revealing properties of the equilibrium before and after the policy is implemented. An equilibrium is revealing if by observing the current and past realized state variables x_t is possible to recover the fundamental disturbances w_t. If this is not the case the equilibrium is said to be non-revealing. Here we present an example showing that non-revealing equilibria are possible and that a feedback control on state variables is capable of turning a non-revealing equilibrium into a revealing one and vice-versa.

We consider univariate models for simplicity as the main insights are maintained as one moves to multivariate analogues. Suppose that the univariate law of motion is:

$$x_t = \beta E_t x_{t+1} + A x_{t-1} + BF x_{t-1} + W(L) w_t . \tag{130}$$

We assume that agents knows the disturbance process $W(L)w_t$. In contrast, the policymaker is assumed to observe only the current and past values of the state variable x_t. Under these assumptions the revealing properties of the equilibria matter in understanding constraints on the policymaker. The agents are assumed to observe directly the fundamental disturbances and so any equilibrium is revealing for them by definition. This parallels the assumption on the asymmetry in the information sets of the agents and the econometrician in Hansen and Sargent (1991). Applying the procedure for solving this type of expectational differential equations as reported in Appendix 1, it can be shown that the moving average solution is

$$G^C(L) = \frac{\frac{1}{\lambda_1^C} W\left(\frac{1}{\lambda_1^C}\right) - LW(L)}{\beta\left(1-\lambda_1^C L\right)\left(1-\lambda_2^C L\right)}, \tag{131}$$

where $1/\lambda_1$ and $1/\lambda_2$ are the roots of the characteristic polynomial $\left(\beta - L + (A+BF)L^2\right)$ and uniqueness requires $\left|1/\lambda_1\right| < 1 < \left|1/\lambda_2\right|$. We consider the case of a simple $MA(1)$ for the disturbance process so that:

$$W(L) = 1 + wL \tag{132}$$

where $|w| < 1$. The unique solution of the uncontrolled system ($F = 0$) can thus be written as:

$$G^{NC}(L) = \frac{\frac{1}{\lambda_1^{NC}}\left(1+\frac{w}{\lambda_1^{NC}}\right)\left(1+\frac{\lambda_1^{NC} w}{\lambda_1^{NC}+w} L\right)}{\beta\left(1-\lambda_2^{NC} L\right)}, \tag{133}$$

where uniqueness requires $|\lambda_1| > 1 > |\lambda_2|$. For this solution to be non-revealing we need that

$$\left|\frac{\lambda_1^{NC} w}{\lambda_1^{NC} + w}\right| > 1. \tag{134}$$

For expositional purposes, we use the point estimates of Gali, Gertler and Lopez-Salido (2005) of a model similar to (130): $A = 0.61$ and $\beta = 0.36$, then $\lambda_1 \simeq \frac{9}{5}$ and $\lambda_2 \simeq \frac{9}{10}$ which confirm the uniqueness of the solution. Now, if one takes $1 > |w| > \left|1 + \frac{5}{9} w\right|$, for instance $w = -\frac{4}{5}$, it follows that $\left|\frac{\lambda_1^{NC} w}{\lambda_1^{NC} + w}\right| > 1$ and according to (134) the equilibrium (133) is non-revealing. The policymaker, or the econometrician, will be able to recover from current and past state variables the innovation process[11]:

$$w_t^* = \left(1 + \frac{\lambda_1^{NC} w}{\lambda_1^{NC} + w} L\right) w_t \tag{135}$$

The range of parameter values for λ, given some w, that imply an equilibrium is non-revealing is an open set, which means that even after applying the control the equilibrium remains non-revealing.

Notice that by choosing appropriately the control F it is possible for

$$\left|\frac{\lambda_1^{C} w}{\lambda_1^{C} + w}\right| < 1. \tag{136}$$

[11] The non-invertible polynomial $\left(1 + \frac{\lambda_1^{NC} w}{\lambda_1^{NC} + w} L\right)$ is a Blaschke factor.

In our numerical example, by choosing F so that $BF = -0.3$ one would still have a unique stable solution and, in addition, a revealing equilibrium since:

$$\left| \frac{\lambda_1^C w}{\lambda_1^C + w} \right| = 0.96 < 1. \tag{137}$$

By the same logic it is possible to show that a feedback control can turn a revealing equilibrium into a non-revealing one.

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