A Recursive Equilibrium Model with Credit Scoring and Competitive Pricing of Default Risk

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Abstract

This paper explores how consumption smoothing (via borrowing and lending) works when a person cannot commit to payback a loan. We study an environment where individuals are of two types, with one type having a higher propensity to default on debt obligations. Financial intermediaries cannot directly observe a person's type but make probabilistic assessments of it based on the person's credit history. We interpret the probability assigned to a person not being of the type that defaults more frequently as the person's *credit score*. In a recursive competitive equilibrium, the terms of credit offered by financial intermediaries depend only on the person's credit score, the amount of credit requested and the risk-free rate. Furthermore, a person's choice of how much to borrow or lend, along with the current score, determines the person's *future* credit score. Thus, the framework delivers an integrated theory of the terms of credit and the credit score. The theory seems capable of explaining some features of the data.

1 Introduction

This paper is part of an ongoing attempt to understand how consumption smoothing (via borrowing and lending) works when borrowers cannot commit to pay back their loans. In this kind of environment, a lender's assessment of a person's creditworthiness is an important determinant of the terms of credit offered to that person. In the U.S., lenders use *credit scores* as an index of a creditworthiness. Most credit scores used in the U.S. are produced by the Fair Isaac and Company and are known as FICO scores. FICO scores range between 300 and 850, where the higher the score greater the individual's creditworthiness.[1]

A credit score is generated by a model that statistically correlates the frequency of default with information in a person's credit history. In the case of FICO scores, the scores take into account five categories of information contained in an individual's credit history: payment history (most particularly the presence of adverse public records such as bankruptcy and delinquency), current amounts owed, the length of the person's credit history, new credit granted and the number of types of credits used (such as credit cards, retail accounts, installment loans, mortgages, etc.).[2]

There is clear evidence that FICO scores affect terms of credit offered to an individual. Table 1 provides information on the relationship between FICO score and the average interest rate on a 60-month new auto loan or a 30-year fixed mortgage. Notice that lower a person's creditworthiness (i.e. lower the FICO score), the higher the interest rate. Two related observations were made by Musto (1999) when he studied the impact of striking an individual's bankruptcy status from his or her credit history after 10 years (as required by the Fair Credit and Reporting Act) on the individual's aquisition of credit. He found (p.15) "a bankruptcy filing significantly constrains unsecured financing for the entire period when potential creditors can observe it on credit files. Among filers, the better credits acquire substantial new access to unsecured credit when the bankruptcy finally leaves their files, and

[1]According to one source, over 75% of mortgage lenders and 80% of the largest financial institutions use FICO scores in their evaluation and approvals process for credit applications.

[2]The law prohibits credit scores from considering race, color, national origin, sex, and marital status and FICO scores do not consider age, assets, salary, occupation, and employment history.

the concurrent dynamics of credit scores indicate that this reflects a big upward boost in estimated creditworthiness".

Table 1

FICO Score	Auto Loan	Mortgage
720-850	4.94%	5.55%
700-719	5.67%	5.68%
675-699	7.91%	6.21%
620-674	10.84%	7.36%
560-619	15.14%	8.53%
500-559	18.60%	9.29%

Source: http://www.myfico.com/myfico/Credit Central/LoanRates.asp

This paper extends our earlier study (Chatterjee, et.al. 2003) on unsecured consumer credit with risk of default to include a simple model of credit scores. We consider an environment where people may be one of two types and a person's type is private information.[3] The two types differ in terms of their attitude to risk, their degree of impatience and the riskiness of their earnings stream. Because these differences obviously bear on the willingness of each type of person to honor his or her liabilities, intermediaries must form some assessment of a person's type. We model this as a *recursive* Bayesian inference problem: intermediaries use a person's current actions in the credit market to update their prior probability of the person being of a given type. If one type of individual always has a lower probability of default (*ceteris paribus*), we can interpret the probability of a person being of that (good) type as the person's credit score.

In our earlier work an individual's credit history could be either good or bad (bankruptcy on file or not) and a defaulting individual was constrained from borrowing for a period of (on average) 10 years – an assumption broadly consistent with Musto's (1999) findings. In this paper, a person's credit history is summarized by a credit score. In equilibrium, adverse developments in the credit market – such as default on a previous loan or increased borrowing – leads to less probability being put on the person being of the good type. This, in turn,

[3]Evidence for heterogeneity in individual type is provided in Ausubel (1999).

results in an adverse change in the terms of credit which might involve a denial of credit altogether. In other words, this paper takes a step toward *explaining* Musto 's finding rather than taking the finding as given.

2 Model Economy

We begin by describing the market arrangement in our model economy. This is followed by a recursive formulation of the individual's decision problem and a description of profit maximizing behavior of firms serving the unsecured credit industry.

2.1 Default Option and Market Arrangement

We model the default option to resemble, in procedure, a Chapter 7 bankruptcy filing. If an individual files for bankruptcy, the individual's beginning of period liabilities are set to zero (i.e., its debts are discharged) and the individual is not permitted to save in the filing period.

There is a competitive credit industry that accepts deposits and makes loans to individuals. An individual can borrow at an interest rate that depends on the total amount borrowed and on the market's belief about the individual's type. Belief about an individual's type is important because an individual cannot commit to repay his or her loan and probability of repayment can vary across types. An individual can also save via deposits and all deposits fetch a constant risk-free rate.

Let $\ell \in L \subset R$ be an individual's asset holding, where $\ell < 0$ denotes debt and $\ell \geq 0$ denotes either storage or loans to other individuals (in equilibrium loans will be risk-free and will fetch a return at least as large as $(1 + r)$). The set L is taken to be finite. The price of a loan of size ℓ made to an individual whose probability of being type g is s is denoted $q(\ell, s) > 0$.

We will refer to s, the market's assessment of an individual's probability of being of type g, as the individual's *credit score.* A key feature of the market arrangement is that an individual's credit score can change according to the actions taken by the individual in the asset market. An individual with loan of size ℓ and a current score s will have score s' next period, where

$$s' = \Psi(\ell', d, s, \ell).$$

where ℓ' is the individual's choice of asset holdings and d is an indicator variable that takes on the value of 1 if the individual defaults on its current loan ℓ and zero otherwise. The functions $\{q, \Psi\}$ are equilibrium objects.

2.2 People

Time is discrete and indexed by t. There are two types of individuals, indexed by $i = g, b$. Each period a unit measure of individuals are born into this economy and of this unit measure a fraction $\gamma \in (0,1)$ is of type g and a fraction $1-\gamma$ is of type b. An individual's type is private information. An individual dies with a constant probability $(1-\delta) > 0$.

Each individual receives a random endowment of goods each period of life. For individuals of type i, this endowment is an i.i.d. draw of a random variable with measure μ^i on a compact support $E = [\underline{e}, \overline{e}] \subset R_{++}$.

The state variables for the individual is the quadruplet (e, ℓ, s). Denote the value function of an individual of type i by $v^i(e, \ell, s)$. Then, the value function solves the following functional equation.

Case 1: When $\ell < 0$

$$v^i(e,\ell,s) = \max \quad \{v^i_1(e,\ell,s), v^i_0(e,\ell,s)\} \tag{1}$$

where

$$v^i_1(e,\ell,s) = u^i(e) + \beta^i \delta \int_E v^i\left[e', 0, \Psi(0,1,s,\ell)\right] \mu^i(de) \tag{2}$$

and

$$v^i_0(e,\ell,s) = \max_{c,\ell'} \; u^i(c) + \beta^i \delta \int_E v^i\left[e', \ell', \Psi(\ell', 0, s, \ell)\right] \mu^i(de) \qquad \text{s.t.} \tag{3}$$

$$(c, \ell') \in B(e, \ell; q) \tag{4}$$

$$\text{where} \qquad B(e,\ell;q) = \{c \geq 0, \ell' \in L \mid c + q(\Psi(\ell', 0, s, \ell), \ell') \cdot \ell' \leq e + \ell\} \tag{5}$$

Here, $u^i(c)$ is the utility that an individual of type i receives from consuming c units of the good and β^i is the discount factor of individual of type i. The value under default, $v_1^i(e,\ell,s)$, assumes that default wipes out all debt and that a defaulting individual cannot accumulate any asset in the period of default. Denote the set of earnings and costs for which an individual of type i and score s will default on a loan of size ℓ by $D^i(\ell,s;q,\Psi) \subseteq E$. The dependence of the default set on q and Ψ is explicitly noted.

Case 2: When $\ell \geq 0$

$$v^i(e,\ell,s) = \max_{c,\ell'} \; u^i(c) + \beta^i \, \delta \int_E v^i \left[e', \ell', \Psi\left(\ell', 0, s, \ell\right)\right] \; \mu^i(de) \qquad \text{s.t.} \tag{6}$$

$$(c,\ell') \in B(e,\ell;q) \tag{7}$$

Let $E^i(\ell,\ell',s;q,\Psi) = \{e \mid \ell^{i\prime}(e,s,\ell) = \ell'\}$ be the set of e for which an individual of type i with score s and loan ℓ chooses ℓ'. Again, the dependence of this set on q and Ψ is explicitly noted.

3 The Credit Industry

Financial intermediaries have access to an international credit market where they can borrow or lend at the risk-free interest rate $r \geq 0$. The profit on a loan of of size $\ell < 0$ is the present discounted value of inflows less the current value of outflows and the profit on deposit of size $\ell > 0$ is the current value of the inflows less the present discounted value of outflows. Therefore, if $a(\ell,s) \geq 0$ is the measure of size ℓ contract (either a loan or a deposit) made by an intermediary, the per-capita expected profits from it is

$$\pi(a(\ell,s;q)) = \begin{cases} a(\ell,s)\cdot(1+r)^{-1}\delta[1-p(\ell,s)](-\ell) - a(\ell,s)\cdot q(\ell,s)(-\ell) & \text{if } \ell < 0 \\ a(\ell,s)\cdot q(\ell,s) - a(\ell,s)\cdot(1+r)^{-1}\delta\cdot\ell & \text{if } \ell \geq 0 \end{cases} \tag{8}$$

where $p(\ell,s)$is the faction of individuals with score s expected to default on a loan of size ℓ. Note that the implicit interest rate on loans takes into account that some individuals will not survive to repay their loans and the implicit interest rate on deposits is also higher to take into account of the fact that some depositors may not survive to collect on their deposits. The full expected profits of an intermediary is then simply $\sum_{\ell} \pi(a(\ell, s; q))$. The decision problem of each intermediary is to maximize full profits subject to the constraint that $a(\ell, s) \geq 0$ for all (ℓ, s).

4 Equilibrium

The equilibrium requirements involve four sets of conditions. The first set are the optimization conditions of individuals. That is, given q and Ψ, $D^i(\ell, s; q, \Psi)$ and $E^i(\ell', \ell, s; q, \Psi)$ must be consistent with (1) and (6)

The second set are zero profit conditions for loans and deposits :

$$q(\ell, s) = \begin{cases} \frac{[1-\mu^g(D^g(\ell,s;q,\Psi)]\cdot s+[1-\mu^b(D^b(\ell,s;q,\Psi)]\cdot(1-s)}{(1+r)} & \ell < 0 \\ \frac{1}{(1+r)} & \ell \geq 0 \end{cases} \tag{9}$$

The third, and most important, set is the formula for updating an individual's credit score. The formula must be consistent with Bayes' rule whenever possible.[4] Recall that according to Bayes' rule, the probability that event A is true given that event B is true is $P(A \cap B)/P(B) = P(A)P(B \mid A)/P(B) = P(A \mid B)$, provided $P(B) > 0$. In Bayesian terminology, $P(A)$ is the prior probability that A is true and $P(A \mid B)$ is the posterior probability that A is true given that B is observed.

Let event A be the event "an individual with score s is of type g." By the definition of s, $P(A) = s$. The event B can be one of two mutually exclusive types. In the first type, B is the event "an individual with score s defaults on loan ℓ." Then, $P(B \mid A) = [\mu^g(D^g(\ell, s; q, \Psi))]$, $P(B) = [\mu^g(D^g(\ell, s; q, \Psi))] \cdot s + [\mu^b(D^b(\ell, s; q, \Psi))] \cdot (1-s)$, and the posterior probability that an individual with score s who defaults on a loan of size ℓ is a good type is given by

$$\Psi(0, 1, s, \ell) = \frac{[\mu^g(D^g(\ell, s; q, \Psi))] \cdot s}{[\mu^g(D^g(\ell, s; q, \Psi))] \cdot s + [\mu^b(D^b(\ell, s; q, \Psi))] \cdot (1-s)} \tag{10}$$

[4]This notion of assigning beliefs "whenever possible" as to individual types on the basis of Bayes Rule is similar to what is assumed as part of a definition of Perfect Bayesian Equilibrium (see Fudenberg and Tirole (2000), p. 331-333).

In the second type, B is the event "an individual with score s and loan ℓ chooses ℓ'." Then, $P(B \mid A) = [\mu^g(E^g(\ell,\ell',s;q,\Psi))]$, $P(B) = [\mu^g(E^g(\ell,\ell',s;q,\Psi))] \cdot s + [\mu^b(E^b(\ell,\ell',s;q,\Psi))] \cdot (1-s)$, and the posterior probability that an individual with score s who repays a loan of size ℓ is a good type is given by

$$\Psi(\ell', 0, s, \ell) = \frac{[\mu^g(E^g(\ell,\ell',s;q,\Psi))] \cdot s}{[\mu^g(E^g(\ell,\ell',s;q,\Psi))] \cdot s + [\mu^b(E^b(\ell,\ell',s;q,\Psi))] \cdot (1-s)}$$

Bayes' rule is applicable if the conditioning event has positive probability. The final set of conditions involves assigning values to $\Psi(\ell', d, s, \ell)$ in some reasonable fashion if the conditioning sets are empty. The approach we follow is to assign values that are the same as that for the closest $q(\ell, s)$ for which the conditioning set is non-empty. We illustrate this approach in the example of the following section.

5 Example

5.1 A Simplified Environment

We consider a simple version of the above environment. There are two periods $t = 0, 1$. We will associate an individual's type with his or her preferences. An individual's type is private information. In period $t = 0$, individuals are endowed with observable assets or liabilities $\ell_{t=0} \in L$. We assume that these assets or liabilities distributed independently of a person's type. In periods $t = 0, 1$, an individual receives an *i.i.d.* earnings shock e from a uniform distribution with support $[e_L, e_H]$. There is also a storage technology: one unit of the good at t becomes $1 + r \geq 1$ units of the good at $t + 1$.

In period $t = 0$, the intermediaries accept deposits from individuals that wish to save for next period ($\ell_1 > 0$) and lend to individuals that wish to borrow ($\ell_1 < 0$) where $\ell_{t=1} \in L$.

Let s_1 denote the (intermediation) industry's assessment of an individual's probability of being of type g, where this assessment is formed in $t = 0$ taking into account the individual's decision to default on ℓ_0 or the individual's choice of assets. Obviously, the industry prior is given by the actual fraction of good type individuals (i.e. $s_0 = \gamma$). We call s_1 the individual's *credit score*. The time $t = 0$ price of a loan of size ℓ_1 made to an individual with score s_1 is denoted $q(\ell_1, s_1) \geq 0$. An individual with loan of size ℓ_0 and score s_0 will have score s_1 next period, where

$$s_1 = \Psi(\ell_1, d_0, s_0, \ell_0).$$

where d_t is an indicator variable that takes on the value of 1 if the individual defaults on its loan ℓ and zero otherwise. The functions $\{q, \Psi\}$ are equilibrium objects.

Individuals can default on debt at $t = 0, 1$. If they default at time $t = 0$ (or 1), ℓ_0 (or ℓ_1) is discharged but during the bankruptcy period they cannot save or borrow.

The timing of events in any given period t are as follows: earnings are revealed, if applicable the default decision is made, if applicable assets are chosen and, finally, consumption takes place.

An individual's budget set is thus given by

$$c_0 = \begin{cases} e_0 + \ell_0 - q(\ell_1, s_1)\ell_1 & \text{if } d_0 = 0 \\ e_0 & \text{if } d_0 = 1 \end{cases}$$

$$c_1 = \begin{cases} e_1 + \ell_1 & \text{if } d_1 = 0 \\ e_1 & \text{if } d_1 = 1 \end{cases}$$

where c_t denote an individual's consumption in period t.

Preferences of an individual of type i are given by

$$E \sum_{t=0}^{1} \beta^t \left[u^i(c_t) - \sigma d_t \right]$$

where u^i is a strictly increasing function, $\beta(1 + r) < 1$, $\delta = 0$, and we have assumed – for this finite-horizon problem – that there is a non-pecuniary cost to defaulting.[5] Finally, we assume that $u^b(c_t) = c_t$ and $u^g(c_t) = \ln(c_t)$.

5.2 Constructing an Equilibrium

Given the above environment, we now turn to constructing an equilibrium. As suggested in the preceding section, we will restrict $L = \{-a, 0, a\}$ and choose a parameterization of the

[5] Gross and Souleles (2002) found that after controlling for economic fundamentals, the propensity to default signficantly rose between 1995-1997. They associated this rise with a decline in default costs, both non-pecuniary (stigma) and pecuniary (the consequences of a bad reputation).

economy such that the relevant sets $D^i(\ell_0, \gamma; q, \Psi)$ and $E^i(\ell_1, \ell_0, \gamma; q, \Psi)$ are non-empty. To this end, we will conjecture that individuals behave in the following way, and then verify that our conjecture is correct.

$t = 1$ decisions

ℓ_1	type	default	no default
$-a$	b	$[e_L, e_H]$	$\varnothing$
$-a$	g	$[e_L, e_1^*]$	(e_1^*, e_H)

$t = 0$ decisions

ℓ_0	type	default	$\ell_1 = -a$	$\ell_1 = 0$	$\ell_1 = a$
$-a$	b	$[e_L, e_H]$	$\varnothing$	$\varnothing$	$\varnothing$
$-a$	g	$[e_L, e_0^*]$	low $e \backslash [e_L, e_0^*]$	mid e	high e
0	b	x	$[e_L, e_H]$	$\varnothing$	$\varnothing$
0	g	x	low e	mid e	high e
a	b	x	$\varnothing$	$[e_L, e_H]$	$\varnothing$
a	g	x	$\varnothing$	low/mid e	mid/high e

5.2.1 t=1 behavior

The only decision to be made at $t = 1$ is whether or not to default on a loan of size $\ell_1 = -a$ after the individual receives its earnings shock e_1. This generates a simple cutoff rule; default if $u^i(e_1 + \ell_1) \leq u^i(e_1) - \sigma$ or $\sigma \leq u^i(e_1) - u^i(e_1 + \ell_1)$. With linear preferences this is simply if $\ell_1 \leq -\sigma$ (i.e. default if liabilities (which are negative) are sufficiently high). With log preferences, this is simply $\sigma \leq \ln\left(\frac{e_1}{e_1+\ell_1}\right)$ or $\exp \sigma \leq \frac{e_1}{e_1+\ell_1}$ or $(e_1 + \ell_1) \exp \sigma \leq e_1$ since $c_1 > 0$ given that $u_c^g(0) = \infty$ or $\ell_1 \leq \frac{-e_1(\exp \sigma - 1)}{\exp \sigma}$ (i.e. for a given level of earnings, default if liabilities are sufficiently high and for a given level of liabilities, do not default if earnings are sufficiently high). In particular, since the right hand side is strictly decreasing in e_1, for a given level of liabilities ℓ_1, there exists a cutoff level of e_1, call it e_1^*, such that for $e > e_1^*(\ell_1)$, the individual does not default. Furthermore, e_1^* is decreasing in ℓ_1 (i.e. the lower are liabilities, the less likely the individual is to default). Since the default decision for the type g (risk averse) individual depends on its earnings, we can always find an (e_1, σ) pair such that some type g individuals default while others do not.

5.2.2 t=0 behavior

In period $t = 0$, the individual must choose whether or not to default (d_0) on its inherited liabilities $\ell_0 = -a$ and, conditional upon not defaulting, choose ℓ_1. Let the indirect utility of a type i individual conditional upon d_0 from state (e_0, ℓ_0, s_0) in period $t = 0$ be denoted $v_0^{i,d}(e_0, \ell_0, s_0; q, \Psi)$.

If the individual defaults at $t = 0$, then (since the individual is not allowed to borrow or save during the default period) $\ell_1 = 0$,) the indirect utility is given simply by:

$$v^{i,d_0=1}(e_0, \ell_0, s_0; q, \Psi) = u^i(e_0) - \sigma + \beta \int_E u^i(e_1)\mu(de_1).$$

For our particular preferences,

$$v^{b,d_0=1}(e_0, \ell_0, s_0; q, \Psi) = e_0 - \sigma + \beta \left[\frac{e_H + e_L}{2}\right]$$

and

$$v^{g,d_0=1}(e_0, \ell_0, s_0; q, \Psi) = \ln(e_0) - \sigma + \beta \left[\frac{e_H \ln(e_H) - e_L \ln(e_L)}{e_H - e_L} - 1\right].$$

If a type g individual does not default, the individual's indirect utility is given by:

$$v^{g,d_0=0}(e_0, \ell_0, s_0; q, \Psi) = \max_{\ell_1 \in L} \ln\left(e_0 + \ell_0 - q(\ell_1, s_1)\ell_1\right)$$
$$+\beta \left\{ \int_{e_L}^{e_1^*(\ell_1)} [\ln(e_1) - \sigma] + \int_{e_1^*(\ell_1)}^{e_H} \ln(e_1 + \ell_1) \right\} \mu(de_1).$$

Under the conjecture that all type b individuals default at $t = 1$

$$v^{b,d_0=0}(e_0, \ell_0, s_0; q, \Psi) = \max_{\ell_1 \in L} e_0 + \ell_0 - q(\ell)1, s_1)\ell_1 + \beta \left[\frac{e_H + e_L}{2} + \ell_1(1 - \iota(\ell_1 < 0))\right] - \sigma\iota(\ell_1 < 0).$$

As before, let $E^i(\ell_1, \ell_0, s_0; q, \Psi) \subset E$ denote the set of earnings for which an individual of type i and score s_0 repays ℓ_0 and chooses ℓ_1. Similarly, let $D^i(\ell_t, s_t) \subset E$ denote the set of earnings for which an individual of type i and score s_t defaults on a loan of size ℓ_t. The posterior of an individual's type who does not default and chooses ℓ_1 (i.e. its credit score s_1) is given by

$$\Psi\left(\ell_1, 0, s_0=\gamma, \ell_0\right)=\frac{\left[\mu(E^g(\ell_1, \ell_0, \gamma; q, \Psi))\right]\cdot\gamma}{\left[\mu(E^g(\ell_1, \ell_0, \gamma; q, \Psi))\right]\cdot\gamma+\left[\mu(E^b(\ell_1, \ell_0, \gamma; q, \Psi))\right]\cdot(1-\gamma)}$$

where μ is understood to be the length of the interval of E^i since we are assuming a common uniform distribution. Similarly, the posterior of an individual's type who defaults on a loan of size ℓ_t with score s_t is given by

$$\Psi\left(0, 1, s_t, \ell_t\right)=\frac{\left[\mu^g(D^g(\ell_t, s_t; q, \Psi))\right]\cdot s_t}{\left[\mu^g(D^g(\ell_t, s_t; q, \Psi))\right]\cdot s_t+\left[\mu^b(D^b(\ell_t, s_t; q, \Psi))\right]\cdot(1-s_t)}$$

Finally, for the price function menu to be consistent with equilibrium, we need $q(\ell_1, s_1)=(1+r)^{-1}$ for $\ell_1\geq 0$ and for $\ell_1<0$

$$q(\ell_1, s_1)=(1+r)^{-1}\left\{\left[1-\mu(D^g(\ell_1, s_1; q, \Psi))\right]\cdot s_1+\left[1-\mu(D^b(\ell_1, s_1; q, \Psi))\right]\cdot(1-s_1)\right\}.$$

But $D^g(\ell_1, s_1)=e_1^*(\ell_1)-e_L$ where $e_1^*(\ell_1)=\frac{a\exp\sigma}{(\exp\sigma-1)}$ and $D^b(\ell_1, s_1)=\varnothing$. This implies

$$q_1(-a, s_1)=(1+r)^{-1}\left[1-\frac{\left(\frac{a\exp\sigma}{(\exp\sigma-1)}-e_L\right)}{e_H-e_L}\right]\cdot s_1 \tag{11}$$

so that prices rise linearly with the credit score.

How do we compute s_1? For the case of default at $t=0$, consistent with the above conjecture where the low earning type g individuals default (i.e. those in $(e_0^*(-a)-e_L)$) and all type b individuals default,

$$\Psi\left(0, 1, \gamma, -a\right)=\frac{\left[\frac{e_0^*(-a)-e_L}{e_H-e_L}\right]\cdot\gamma}{\left[\frac{e_0^*(-a)-e_L}{e_H-e_L}\right]\cdot\gamma+1\cdot(1-\gamma)}$$

For the case of no default at $t=0$, there are 9 possible credit score transitions that we must find.

state ↓/ choice →	$\ell_1=-a$	$\ell_1=0$	$\ell_1=a$
$\ell_0=-a$	$\Psi\left(-a, 0, \gamma, -a\right)=1$	$\Psi\left(0, 0, \gamma, -a\right)=1$	$\Psi\left(a, 0, \gamma, -a\right)=1$
$\ell_0=0$	$\Psi\left(-a, 0, \gamma, 0\right)=\frac{\left[\frac{e_0^*(0)-e_L}{e_H-e_L}\right]\cdot\gamma}{\left[\frac{e_0^*(0)-e_L}{e_H-e_L}\right]\cdot\gamma+1\cdot(1-\gamma)}<1$	$\Psi\left(0, 0, \gamma, 0\right)=1$	$\Psi\left(a, 0, \gamma, 0\right)=1$
$\ell_0=a$	$\Psi\left(-a, 0, \gamma, a\right)=0$	$\Psi\left(0, 0, \gamma, a\right)<1$	$\Psi\left(a, 0, \gamma, a\right)=1$

The interesting part of this table, is the first column. Why is $\Psi(-a, 0, \gamma, -a) = 1$? At $\ell_0 = -a$, all type b default and if type g do not, then we know their type. Why is $\Psi(-a, 0, \gamma, 0) < 1$? At $\ell_0 = 0$, all type b choose $-a$ and if type g choose $-a$, the set of those g doing so is given by $e_0^*(0) - e_L$. In that case, we can't tell those g from b. Why does $\Psi(-a, 0, \gamma, a) = 0$? At $\ell_0 = a$, for almost all prices, type b would choose $\ell_1 = -a$ while type g choose either 0 or a. But in that case the intermediary knows that an individual choosing $\ell_1 = -a$ from $\ell_0 = a$ must be a bad type who will default. Thus, it is consistent with an equilibrium to set $\Psi(-a, 0, \gamma, a) = 0$. The only other case where there is some confusion as to type occurs where $\ell_0 = a$ and $\ell_1 = 0$ since some low type g decide to consumption smooth and can't be sorted from the type b. Since we have just established that $\Psi(-a, 0, \gamma, \ell_0)$ is declining in ℓ_0, we know that prices $q(-a, \Psi(-a, 0, \gamma, \ell_0))$ are declining in ℓ_0. This results in an extremely interesting result; a person with past debt gets a better interest rate than a person without any debt (i.e. either 0 or a).

To finish the construction of the equilibrium, we need to check that individuals take the actions we assigned in the above table. While there are some results which we can prove to rule out some behavior, here we will simply choose some parameter values and compute an equilibrium (so we have an equilibrium, but there's no sense in which we know this is the only equilibrium).

Let the parameter values be given by: $\beta = 0.9$, $r = 0.05$, $\gamma = 0.98$, $a = 1.95$, $\sigma = 0.6$, $e_L = 2$, $e_H = 12$. Then

Equilibrium Actions:

$t = 1$ decisions

ℓ_1	type	default	no default
$-a$	b	$[e_L, e_H]$	$\varnothing$
$-a$	g	$[e_L, 4.32]$	$(4.32, e_H)$

$t = 0$ decisions

ℓ_0	type	default	$\ell_1 = -a$	$\ell_1 = 0$	$\ell_1 = a$
$-a$	b	$[e_L, e_H]$	$\varnothing$	$\varnothing$	$\varnothing$
$-a$	g	$[e_L, 2.3]$	$(2.3, 5.5]$	$(5.5, 10.1]$	$(10.1, e_H]$
0	b	x	$[e_L, e_H]$	$\varnothing$	$\varnothing$
0	g	x	$[e_L, 2.9]$	$(2.9, 8.1]$	$(8.1, e_H]$
a	b	x	$\varnothing$	$[e_L, e_H]$	$\varnothing$
a	g	x	$\varnothing$	$[e_L, 6.1]$	$(6.1, e_H]$

Credit Scores: No $t = 0$ Default

$\Psi(\ell_1, 0, \gamma, \ell_0)$	$\ell_1 = -a$	$\ell_1 = 0$	$\ell_1 = a$
$\ell_0 = -a$	1	1	1
$\ell_0 = 0$	0.82	1	1
$\ell_0 = a$	0^*	0.95	1

* Any $\Psi(\ell_1, 0, \gamma, \ell_0) \leq 0.37$ is consistent with equilibrium

Prices

$q(\ell_1, \Psi(\ell_1, 0, \gamma, \ell_0))$	$\ell_1 = -a$	$\ell_1 = 0$	$\ell_1 = a$
$\ell_0 = -a$	0.73	0.95	0.95
$\ell_0 = 0$	0.60	0.95	0.95
$\ell_0 = a$	0^*	0.95	0.95

* Any $q(\ell_1, \Psi(\ell_1, 0, \gamma, \ell_0)) \leq 0.28$ is consistent with equilibrium

To summarize, there are 2 important points that this example illustrates. First, equation (11) implies that borrowing prices are linear in an individual's credit score. The more likely the individual is a good type (i.e. the higher is its score), the higher the price (the lower the interest rate). This is consistent with Table 1 in the Introduction. Second, since payment history is an important determinant of the Bayesian update of an agent's type (that is, $\Psi(\ell', d, s, \ell)$ depends not only on what one chooses to borrow ℓ', but that the individual chose not to default on ℓ), the example shows that individuals who have no history of past borrowing and payback (i.e. those with $\ell_0 \in \{0, a\}$) have a lower credit score than those who have paid back (i.e. those with $\ell_0 = -a$). This is broadly consistent with experience.[6] This translates into either higher costs of borrowing (in the case of $\ell_0 = 0$) or an endogenous credit limit (in the case of $\ell_0 = a$).

To understand the endogenous credit limit, consider the decision problems for type b and g agents with $\ell_0 = a$, respectively. Since $\ell_0 = a$ there is no default decision and all we must consider is the saving/borrowing decision. Considering the type b, $\ell_0 = a$ agent first:

$$v^{b,0}(e_0, a, \gamma; q, \Psi) = \max \left\{ \begin{array}{c} e_0 + a + q_1(-a, \Psi(-a, 0, \gamma, a))a + \beta \left\{ \left[\frac{e_H + e_L}{2}\right] - \sigma \right\}, \\ e_0 + a + \beta \left[\frac{e_H + e_L}{2}\right], \\ e_0 + a - \frac{a}{1+r} + \beta \left\{ \left[\frac{e_H + e_L}{2}\right] + a \right\} \end{array} \right\}.$$

First, notice that the choice of $\ell_1 = 0$ dominates $\ell_1 = a$ since $1 > \beta(1 + r)$ (i.e. the return to storage is too low). This is why there is a $\varnothing$ in the cell for $t = 0$ decisions above. For the choice of $\ell_1 = 0$ to strictly dominate $\ell_1 = -a$, we need $\beta\sigma > q_1(-a, \Psi(-a, 0, \gamma, 0))a$ (i.e. the current gain in consumption from borrowing is lower than the future stigma cost since we know this type will default at $t = 1$). Define $\overline{\Psi}$ by $\beta\sigma = q_1(-a, \overline{\Psi}(-a, 0, \gamma, 0))a$, which is the highest credit score such that a type b agent chooses to borrow (i.e. chooses $\ell_1 = -a$).

[6] As quoted in http://www.myfico.com/myfico/Credit Central/ScoreConsiders/Tips/AmountsOwedTip.asp, "someone with no credit cards, for example, tends to be a higher risk than someone who has managed credit cards responsibly."

Next, consider the type g, $\ell_0 = a$ agent. We have that $v^{g,0}(e_0, a, \gamma; q, \Psi)$ is given by

$$\max \left\{ \begin{array}{c} \ln\left(e_0 + a + q_1(-a, \Psi\left(-a, 0, \gamma, a\right))a\right) + \beta \left\{ \int_{e_L}^{e_1^*(-a)} \left[\ln(e_1) - \sigma\right] + \int_{e_1^*(-a)}^{e_H} \ln(e_1 - a) \right\} \mu(de_1) \\ \ln\left(e_0 + a\right) + \beta \left[\frac{e_H \ln(e_H) - e_L \ln(e_L)}{e_H - e_L} - 1 \right] \\ \ln\left(e_0 + a - \frac{a}{1+r}\right) + \beta \left[\frac{[a+e_H] \ln(a+e_H) - [a+e_L] \ln(a+e_L)}{e_H - e_L} - 1 \right] \end{array} \right\}.$$

For the choice of $\ell_1 = 0$ to strictly dominate $\ell_1 = -a$ we must have

$$\beta \left\{ \int_{e_L}^{e_1^*(-a)} \sigma + \int_{e_1^*(-a)}^{e_H} \left\{ \ln(e_1) - \ln(e_1 - a) \right\} \right\} \mu(de_1) > \ln\left(e_0 + a + q_1(-a, \Psi\left(-a, 0, \gamma, a\right))a\right) - \ln\left(e_0 + a\right) \tag{12}$$

Under what conditions at $\overline{\Psi}$, will a type g agent strictly prefer $\ell_1 = 0$ to strictly dominate $\ell_1 = -a$. At $\overline{\Psi}$, equation (12) can be written

$$\beta \left\{ \int_{e_L}^{e_1^*(-a)} \sigma + \int_{e_1^*(-a)}^{e_H} \left\{ \ln(e_1) - \ln(e_1 - a) \right\} \right\} \mu(de_1) > \ln\left(e_0 + a + \beta\sigma\right) - \ln\left(e_0 + a\right) \tag{13}$$

A sufficient condition for this to hold is

$$\beta \left\{ \frac{\sigma(a - e_L) + (e_H - a)\left[\ln(e_H) - \ln(e_H - a)\right]}{e_H - e_L} \right\} > \ln\left(e_0 + a + \beta\sigma\right) - \ln\left(e_0 + a\right)$$

Notice that the left hand side is decreasing in e_H, so there is a range of values of e_H sufficiently small such that we can always satisfy this condition.

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