

# BOND SUPPLY, EXPECTATIONS, AND THE YIELD CURVE\*

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## Abstract

This paper studies the joint dynamics of the term structure of interest rates and US household holdings of fixed-income securities. It uses an affine term structure model to describe observed household positions by portfolios with a small number of zero-coupon bonds. It then views the evolution of the yield curve as a sequence of temporary equilibria: bond prices are determined to equate households' demand for bonds to the supply of these bonds provided by the rest of the economy. Demand is determined by solving the portfolio problem of a representative household with recursive utility, given expectations about future bond payoffs implied by the affine term structure model.

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# I Introduction

A large literature has tried to jointly model the term structure of interest rates and macroeconomic fundamentals. Recent arbitrage-free models aim to explain the dynamics of the yield curve with a small number of factors that include macroeconomic variables such as inflation and output. Consumption-based asset pricing models link the term structure to consumption growth and inflation, and possibly features of the monetary policy process. However, empirical analysis of both types of models does typically not explore implications for household holdings of nominal assets.

This paper proposes a structural model of the term structure that jointly considers interest rates and US household sector bond positions. We view the US household sector as an infinitely-lived representative agent with recursive utility. The agent can invest in bonds and also in a “residual asset” that stands in for equity, housing and human capital. He can trade assets with the “rest of the economy” (ROE) sector, which follows an exogenously given trading strategy. We identify this ROE trading strategy with observed household sector asset trades. We then look for prices that make households hold the “right” amount of bonds in equilibrium.

More specifically, the empirical implementation of the model proceeds in four steps. First, we estimate a VAR on nominal interest rates, inflation and the return on the residual asset to capture the agent’s expectations about future asset payoffs. Second, we solve the agent’s savings and portfolio choice problem for every period in our sample, given current asset prices as well as asset payoff expectations implied by the VAR. This step gives rise to a household asset demand function for every period. Third, we determine equilibrium asset prices for every period by matching up household asset demand with the trading strategy of the rest of the economy. We thus obtain a sequence of equilibrium prices of the same length as the sample. The model is “successful” if the sequence of equilibrium prices is close to actual prices.

US households hold a large variety of nominal instruments, so that their “bond position” is high-dimensional. To address this issue, we use an arbitrage-free affine term structure model to show that household sector nominal positions in the Flow-of-Funds Accounts (FFA) can be replicated by portfolios consisting of only three zero coupon bonds. Three bonds work because a two factor model does a good job describing the nominal term structure. We use the replicating portfolios

to construct three series of bondholdings, which represent the trading strategy of the rest of the economy. One interesting fact from this step is that the relative supply of longer bonds declined before 1980, as interest rate spreads were falling, but saw a dramatic increase in the 1980s, a time when spreads were extraordinarily high.

For a benchmark empirical exercise, we choose preference parameters to match the mean yield curve, and make households' expectations about the (unobservable) return on the residual consistent with model-implied returns on that asset. We show that this strategy implies a representative household who is patient and not very risk averse (the coefficient of relative risk aversion is 3.5). In equilibrium, bonds make up only a small fraction of total wealth, and the household tries to aggressively time the bond market. As a result, shifts in bond demand due to expectations about future payoffs are larger and occur at higher frequencies than changes in the trading strategy of the rest of the economy. Market clearing requires price adjustments that induce high frequency fluctuations in yield spreads. Equilibrium yields are therefore as nearly as volatile as those in the data and mimic many movements of actual yields, especially at business cycle frequencies.

The rest of the paper is structured as follows. Section II introduces the model. Section III explains how we implement it empirically. In particular, it shows how to replicate nominal positions by simple portfolios. Section IV reports the asset pricing results.

## II Model

A large number of identical households lives forever. Their preferences over consumption plans are represented by Epstein-Zin utility with unitary intertemporal elasticity of substitution. The utility  $u_t$  of a consumption plan  $(C_\tau)_{\tau=t}^\infty$  solves

$$(1) \quad u_t = (1 - \beta) \log C_t + \beta \log E_t \left[ e^{(1-\gamma)u_{t+1}} \right]^{\frac{1}{1-\gamma}}.$$

Households' ranking of certain consumption streams is thus given by discounted logarithmic utility. At the same time, the attitude towards (atemporal lotteries) is determined by the risk aversion coefficient  $\gamma$ . We focus below on the case  $\gamma > 1$ , which implies an aversion to persistent risks (as in

Piazzesi and Schneider 2006).

In every trading period  $t$ , the households trade assets with the rest of the economy (ROE) sector, which stands in for the government, business, and foreign sectors. There are two types of assets. *Bonds* are nominal instruments that promise dollar-denominated payoffs in the future. In particular, there is a one period bond – from now on, the *short* bond – that pays off one dollar at date  $t + 1$ ; it trades at date  $t$  at a price  $e^{-i_t}$ . Its real payoff is  $e^{-\pi_{t+1}}$ , where  $\pi_t$  is (log) inflation.<sup>1</sup>

At every date  $t$ , households can buy  $N$  other zero-coupon bonds, which we refer to as *basic* bonds. We collect the log nominal prices of these bonds at date  $t$  in a vector  $\hat{p}_t$ , and we collect their log nominal payoffs<sup>2</sup> at date  $t + 1$  in a vector  $\hat{p}_{t+1}^{+1}$ . The log excess returns over the short bond from date  $t$  to date  $t + 1$  can thus be written as  $\hat{x}_{t+1} = \hat{p}_{t+1}^{+1} - \hat{p}_t - i_t$ . Below, the number of basic bonds  $n$  will correspond to the number of factors in our term structure model: our empirical implementation will use the fact that, under an  $N$ -factor model,  $N + 1$  bonds are sufficient to span the payoffs on all bonds.

In addition to bonds, households trade a *residual asset*, which stands in for all assets other than bonds. The residual asset pays a dividend  $Y_t$  at date  $t$  and then trades at an ex-dividend price  $P_t^{res}$ . Its log real return from date  $t$  to date  $t + 1$  is  $r_{t+1}^{res}$ , so that its excess return over the short bond is  $x_{t+1}^{res} = r_{t+1}^{res} - i_t - \pi_{t+1}$ . Introducing the residual asset allows for a total supply of assets that is larger than just all outstanding bonds. In practice, the most important components of the residual asset are equity, residential real estate, and human capital. The dividend  $Y_t$  thus includes dividends on equity, housing services, and labor income. Our model will not speak to the relative prices of these component assets, but only relate their aggregate value to that of bonds.

At the beginning of a trading period  $t$ , households are endowed with one unit of the residual asset. Their initial wealth  $\overline{W}_t$  also comprises dividends from the residual asset as well as the value

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<sup>1</sup>This is a simple way to capture that the short bond is denominated in dollars. To see why, consider a nominal bond which costs  $q_t$  dollars today and pays of \$1 tomorrow, or  $1/p_{t+1}^c$  units of numeraire consumption. Now consider a portfolio of  $p_t^c$  nominal bonds. The price of the portfolio is  $q_t$  units of numeraire and its payoff is  $p_t^c/p_{t+1}^c = 1/\pi_{t+1}$  units of numeraire tomorrow. The model thus determines the price  $q_t$  of a nominal bond in \$.

<sup>2</sup>This notation is convenient to accomodate the fact that the maturity of zero-coupon bonds changes from one date to the next. For example, let  $i_t^{(n)}$  denote the yield to maturity on a zero-coupon bond of maturity  $n$ , and assume that there there is only one basic bond, of maturity  $\hat{n}$ . The basic bond trades at date  $t$  at a log price  $\hat{p}_t = -\hat{n}i_t^{(\hat{n})}$ , and it promises a log payoff at date  $t + 1$  of  $\hat{p}_{t+1}^{+1} = -(\hat{n} - 1)i_{t+1}^{(\hat{n}-1)}$ . Our assumptions on the trading strategy of the rest of the economy below will imply that only the basic bonds and the short bond are ever held by households in equilibrium.

of all bonds written on the rest of the economy that households bought at date  $t - 1$ , denoted  $\bar{B}_t$ :

$$(2) \quad \bar{W}_t =: P_t^{res} + Y_t + \bar{B}_t.$$

Households decide how to split this initial wealth into consumption as well as investment in the  $N + 2$  assets. We denote by  $\alpha_t^{res}$  the portfolio weight on the residual asset (that is, the fraction of savings invested in that asset), and we collect the portfolio weights on all bonds other than the short bond in a vector  $\hat{\alpha}_t$ . The household's sequence of budget constraints can then be written as<sup>3</sup>

$$(3) \quad \begin{aligned} \bar{W}_{\tau+1} &= R_{\tau+1}^w (\bar{W}_\tau - C_\tau), \\ R_{\tau+1}^w &= e^{i_\tau - \pi_{\tau+1}} \left( 1 + \alpha_\tau^{res} e^{x_\tau^{res}} + \hat{\alpha}_\tau^\top e^{\hat{x}_{\tau+1}} \right); \quad \tau \geq t. \end{aligned}$$

The household problem at date  $t$  is to maximize utility (1) subject to (3), given initial wealth  $\bar{W}_t$  as well as beliefs about (a conditional distribution of) the relevant future price variables  $(x_\tau^{res}, i_\tau, \pi_\tau, \hat{p}_\tau, \hat{p}_\tau^{+1})_{\tau > t}$ : the excess return on the residual asset, the short interest rate, the inflation rate and the prices and payoffs on the basic bonds.

The trading strategy of the rest of the economy is exogenous. In particular, at date  $t$  the ROE sector sells residual assets worth  $Pf_t$ . This trade captures, for example, the construction of new houses, and the net issuance of new equity. The ROE also trades in the bond market. It is convenient to summarize bond trades in terms of the value of outstanding bonds written on the ROE. At date  $t$ , the ROE redeems all short bonds issued at date  $t - 1$ , and it also buys back all outstanding basic bonds, at a total cost of  $\bar{B}$ . Moreover, the ROE issues new bonds worth  $D_t$ , so that its net sale of bonds is  $D_t - \bar{B}_t$ , which could be positive or negative.

To define new issues of individual bonds, we collect the values of new basic bonds in a vector  $\hat{D}_t$ . The value of outstanding short bonds is then  $D_t - \iota' \hat{D}_t$ , where  $\iota$  is an  $N$ -vector of ones. The ROE trading strategy is thus set up so that the same set of  $N + 1$  types of bonds – namely the short bond and the basic bonds – are held by households at the end of every trading period. For example, suppose that the only basic bond is a zero-coupon bond with a maturity of  $n$  periods. Between dates  $t - 1$  and  $t$ , households can then hold short (1-period) and long ( $n$ -period) bonds.

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<sup>3</sup>Here  $e^{\hat{x}^t}$  is an  $N$ -vector with the  $j$ th element equal to  $e^{\hat{x}^t, j}$ .

At date  $t$ , the ROE buys back all long bonds (which now have maturity  $n - 1$ ), and again issues new 1-period short and  $n$ -period long bonds, and so on.

### *Equilibrium*

We solve for a sequence of temporary equilibria. For each trading date  $t$ , we take as given (i) the strategy of the rest of the economy, summarized by its asset trades  $(Pf_t, \bar{B}_t, D_t, \hat{D}_t)$ , (ii) dividends  $Y_t$  on the residual asset earned by households and (iii) household expectations about (that is, the conditional distribution of) the relevant price variables  $(r_\tau^{res}, i_\tau, \pi_\tau, \hat{p}_\tau, \hat{p}_\tau^{+1})_{\tau > t}$ , which comprise the return on the residual asset, the short interest rate, the inflation rate and the prices and payoffs on the basic bonds for all future periods. We characterize equilibrium prices as functions of these three inputs by equating household asset demand to the net asset supply provided by the trading strategy of the ROE.

Formally, an equilibrium consists of sequences of short interest rates and (log) basic bond prices  $(i_t, \hat{p}_t)$  as well as optimal choices by households  $(C_t, \alpha_t^{res}, \hat{\alpha}_t)$  such that, at every date  $t$ , all four asset markets clear:

$$\begin{aligned}
 \alpha_t^{res} (\bar{W}_t - C_t) &= P_t^{res} + Pf_t; \\
 \hat{\alpha}_t (\bar{W}_t - C_t) &= \hat{D}_t; \\
 (4) \quad \bar{W}_t - C_t &= D_t + P_t^{res} + Pf_t
 \end{aligned}$$

Here the first equation clears the market for the residual asset, the second equation clears the markets for the basic bonds and the last equation ensures that total savings equals the total value of outstanding assets, which implies that the market for short bonds also clears. This system of  $N + 2$  equations determines the  $N + 2$  asset prices  $(P_t^{res}, i_t, \hat{p}_t)$ . While the price of the residual asset  $P_t^{res}$  appears directly in (4), bond prices enter via the effect of bond returns on portfolio demand.

A sequence of temporary equilibria imposes weaker restrictions on allocations and prices than a standard rational expectations equilibrium. In particular, the definition above does not directly connect what happens at different trading periods. On the one hand, we do not require that the initial wealth of households is derived from its choices in the previous period. For example,

the sequence  $(\bar{B}_t)$  of payoffs from bonds bought earlier is an exogenous input to the model. On the other hand, we do not impose conditions relating return expectations at date  $t$  to model-implied realized (or expected) returns in future periods, as one would do when imposing rational expectations. At the same time, if there is a rational expectations equilibrium of our model that accounts for observed asset prices and household sector choices, then it also gives rise to a sequence of temporary equilibria. When we explain our empirical strategy below, we will argue that imposing weaker theoretical restrictions on asset positions and expectations is useful when implementing a possibly misspecified model.

The fact that our model allows for trades between the household sector and the rest of the economy distinguishes it from the endowment economies frequently studied in the asset pricing literature. In particular, our model accommodates nonzero personal savings. Combining (2) and the last equation in (4), we obtain the flow-of-funds identity

$$(5) \quad C_t + Pf_t + (D_t - \bar{B}_t) = Y_t.$$

The dividend on the residual asset  $Y_t$  corresponds to personal income less net personal interest income. As a result,  $Y_t - C_t$  is personal savings less net interest. It consists of  $Pf_t$  – net purchases of all assets other than bonds – and  $D_t - \bar{B}_t$  – net purchases of bonds less net interest. A typical endowment economy model of the type studied by Lucas (1978) instead assumes that bonds are in zero net supply and that household net wealth is a claim on future consumption, so that  $C_t = Y_t$  and  $Pf_t = 0$ .

### *Savings, Portfolio Choice and Asset Prices*

We now derive approximate formulas for temporary equilibrium asset prices as functions of the three model inputs – the ROE strategy, dividends and beliefs. Beliefs are described as follows. Let  $z_t$  denote a vector of exogenous state variables that follows a stationary vector autoregression with homoskedastic normally distributed shocks. We assume that log short interest rates  $i_t$  and log inflation  $\pi_t$  are time invariant linear functions of  $z_t$  and that log excess returns  $x_t = (x_t^{res}, \hat{x}_t)'$  are time-invariant linear functions of  $z_t$  and  $z_{t-1}$ . Households' belief about future returns, interest rates and inflation at date  $t$  is now defined as the conditional implied by the VAR. As a result, the

state vector for the household problem is  $(\bar{W}_t, z_t)$ .

The price of the residual asset is determined independently of return expectations. Indeed, under the preferences (1), the value function takes the form  $v(\bar{W}_t, z_t) = \log \bar{W}_t + \tilde{v}(z_t)$  and the household savings rate out of initial wealth  $\bar{W}_t$  is equal to the discount factor, that is,  $\bar{W}_t - C_t = \beta \bar{W}_t$ .<sup>4</sup> Using this expression for savings in the last equation of (4), and substituting for  $\bar{W}_t$  from (2) yields

$$(6) \quad P_t^{res} = \frac{1}{1-\beta} (\beta (Y_t + \bar{B}_t) - D_t - Pf_t).$$

In other words, given the preference parameter  $\beta$ , a series for  $P^{res}$  can be derived from the exogenous inputs  $(Y_t, D_t, \bar{B}_t, Pf_t)$ .

To derive bond prices, we must consider how the optimal portfolio weights  $\alpha_t = (\alpha_t^{res}, \hat{\alpha}_t)'$  depend on all inputs, including beliefs. Under our assumptions on beliefs, we can use the approximation method proposed by Campbell, Chan and Viceira (2002). The basic idea is that the log return on a portfolio in a discrete time problem is well approximated by a discretized version of its continuous time counterpart. In our setup, the log return on wealth is approximated by

$$(7) \quad \log R_{t+1}^w \approx i_t - \pi_{t+1} + \alpha_t' x_{t+1} + \frac{1}{2} \alpha_t' (\text{diag}(\Sigma_{xx}) - \Sigma_{xx} \alpha_t),$$

where  $\Sigma_{xx}$  is the one-step-ahead conditional covariance matrix of  $x_{t+1}$ .

Following Campbell et al., the optimal portfolio is

$$(8) \quad \alpha_t \approx \frac{1}{\gamma} \Sigma_{xx}^{-1} \left( E_t [x_{t+1}] + \frac{1}{2} \text{diag}(\Sigma_{xx}) \right) + \left( 1 - \frac{1}{\gamma} \right) \Sigma_{xx}^{-1} \text{cov}_t(x_{t+1}, \pi_{t+1}) - \left( 1 - \frac{1}{\gamma} \right) \Sigma_{xx}^{-1} \text{cov}_t(x_{t+1}, \tilde{v}(z_{t+1})).$$

If  $\gamma = 1$  – the case of separable logarithmic utility – the household behaves “myopically”, that is, the portfolio composition depends only on the one-step-ahead distribution of returns. More generally, the first line in (8) represents the myopic demand of an investor with one-period horizon and risk aversion coefficient  $\gamma$ . To obtain intuition, consider the case of independent returns, so

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<sup>4</sup>This is due to the unitary intertemporal elasticity of substitution. As far as the consumption-savings decision is concerned, the model thus behaves like one with separable log utility. However, the portfolio choice decision is different, as will become clear below.

that  $\Sigma_{xx}$  is diagonal. The first term then says that the myopic investor puts more weight on assets with high expected returns and low variance, and more so when risk aversion is lower. The second term says that, if  $\gamma > 1$ , the investor also likes assets that provide insurance against inflation, and buys more such insurance assets if risk aversion is higher. For general  $\Sigma_{xx}$ , these statements must be modified to take into account correlation patterns among the individual assets.

For a long-lived household with  $\gamma \neq 1$ , asset demand also depends on the covariance of excess returns and future continuation utility  $\tilde{v}(z_{t+1})$ . Continuation utility is driven by changes in investment opportunities: a realization of  $z_{t+1}$  that increases  $\tilde{v}$  is one that signals high returns on wealth (“good investment opportunities”) in the future. Agents with  $\gamma > 1$  prefer relatively more asset payoff in states of the world where investment opportunities are bad. As a result, an asset that pays off when investment opportunities are bad is unattractive for a high- $\gamma$  agent. He will thus demand less of it than a myopic agent.

We want to solve the first  $N + 1$  market clearing conditions in (4) for the  $N + 1$  bond prices. It is helpful to divide by  $\bar{W} - C_t = \beta\bar{W}_t$ , so that the right hand side in every equation depends only on the ROE strategy, dividends and the discount factor. Letting  $\delta = \frac{1-\beta}{\beta}$  denote the rate of time preference, we define

$$(9) \quad \alpha_t^W := \begin{pmatrix} 1 - \delta \frac{D_t}{Y_t + B_t - D_t - Pf_t} \\ \delta \frac{\hat{D}_t}{Y_t + B_t - D_t - Pf_t} \end{pmatrix} = \begin{pmatrix} 1 - \delta D_t / C_t \\ \delta \hat{D}_t / C_t \end{pmatrix},$$

where the second equality uses (5). In equilibrium, bond prices must adjust to make household portfolio demand (8) equal to  $\alpha_t^M$ .<sup>5</sup> Since the model implies a fixed ratio of consumption to wealth, changes in the total quantity of outstanding bonds relative to consumption affect the equilibrium portfolio share on the residual asset. The shares on individual bonds depend in addition on how much of each bonds is sold to households by the rest of the economy.

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<sup>5</sup>The vector  $\alpha_t^M$  represents a “market portfolio” that collects the fraction of total savings invested in the residual asset and the basic bonds. We emphasize that the market portfolio in our model is not entirely exogenous. This is because we have already solved out for the price of the residual asset.

Combining (8) and (9), the equilibrium risk premia are

$$\begin{aligned}
 E_t[x_{t+1}] + \frac{1}{2}diag(\Sigma_{xx}) &= \gamma\Sigma_{xx}\alpha_t^W + (1 - \gamma)cov_t(x_{t+1}, \pi_{t+1}) + (\gamma - 1)cov_t(x_{t+1}, \tilde{v}(z_{t+1})) \\
 (10) \qquad \qquad \qquad &= : \gamma\Sigma_{xx}\alpha_t^W + \rho_t.
 \end{aligned}$$

We follow the convention in the literature to add one half the variance to the expected log excess return as a correction for Jensen’s inequality. In addition,  $\rho_t = (\rho_t^{res}, \hat{\rho}_t)'$  is a vector that stacks the relevant covariances for the residual assets and the basic bonds.

Every risk premium consists of three parts. The first is proportional to  $\Sigma_{xx}\alpha_t^W$ , the covariance of excess returns with the (equilibrium) return on wealth  $x'_{t+1}\alpha_t^W$ . In the case of log utility ( $\gamma = 1$ ) this covariance represents the entire risk premium. Log returns then follow a conditional version of the Capital Asset Pricing Model (CAPM): the risk premium on an asset is higher the more that asset covaries with the “market portfolio”  $\alpha^W$ . More generally, a higher risk aversion coefficient drives up the compensation required for covariance with the return on wealth.<sup>6</sup> Since the composition of wealth changes over time, for example with changes in the relative amount of different bonds in  $\hat{D}_t$ , there is time variation in risk premia.

The second term is a negative inflation risk premium. For  $\gamma > 1$ , this premium is negative: households have to be compensated less to hold an asset that provides insurance against inflation. Since our VAR is heteroskedastic, the inflation risk premium is constant over time. Finally, the third term is a premium for covariance with future investment opportunities. An asset that insures households against bad future investment opportunities – by paying off less when continuation utility  $\tilde{v}$  is high - commands a lower premium. For our numerical results below, the function  $\tilde{v}$  will be approximately quadratic in  $z_{t+1}$ . This implies that insurance against bad opportunities is another source of time-variation in risk premia.

Substituting the definitions of the excess returns  $x_t^{res}$  and  $\hat{x}_t$  into the first equation in (10), we

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<sup>6</sup>Since consumption is proportional to savings, or wealth, the first term in the risk premium also represents the covariance of returns with consumption growth, multiplied by risk aversion. In the log case, the CAPM is equivalent to the consumption CAPM.

obtain the equilibrium short rate

$$i_t = (E_t r_{t+1}^{res} - \gamma cov_t(x_{t+1}^{res}, x_{t+1}) \alpha_t^W - \rho_t^{res}) + E_t \pi_{t+1} - \frac{1}{2} var_t \pi_{t+1}.$$

Intuitively, expectations about the real return on the residual asset and the risk premium on that asset fixes the real interest rate. In addition, expected inflation has a direct effect on the nominal rate.

To write out the equilibrium prices of the basic bonds, it is helpful to define  $\Sigma_{\hat{p}\hat{p}}$  as the covariance matrix of next period's basic bond payoffs (the lower right  $N$ -by- $N$  block of  $\Sigma_{xx}$ ). We can then rearrange the last  $N$  lines in (10) as

$$\hat{p}_t = E_t \hat{p}_{t+1}^+ - i_t \iota - \left[ \gamma cov_t(\hat{p}_{t+1}^+, x_{t+1}) \alpha_t^W + \hat{\rho}_t + \frac{1}{2} diag(\Sigma_{\hat{p}\hat{p}}) \right]$$

where  $\iota$  is an  $N$ -vector of ones. The term in square brackets contains the risk premia on the basic bonds, up to the correction for Jensen's inequality. If the risk premium is constant, a version of the expectations hypothesis holds: on average up to a constant risk premium, buying a long bond at  $t$  and holding it to maturity should cost the same as buying a short bond at  $t$ , earning interest  $i_t$  on it and then buying the long bond only at  $t + 1$ .

### III Empirical Implementation

In this section, we describe how we quantify the three exogenous inputs to our model – the trading strategy of the rest of the economy, the dividend on the residual asset and household expectations about future prices and returns. In subsection A., we use an affine term structure model to argue that a setup with only three bonds—a short bond and two basic bonds—does a good job in capturing the rich set of fixed income instruments held by US households. In subsection B., we map Flow-of-Funds data on household positions in fixed income securities into time series of positions in the three bonds. This series represents the bond trading strategy of the rest of the economy. In subsection C., we also derive the ROE's trading in the residual asset from Flow-of-Funds data. Finally, subsection D. uses our term structure model together with data on inflation and the model-implied return on

the residual asset to specify the VAR for expectations.

## A. Bonds

Interest rates on bonds of different maturities are highly correlated. Motivated by this fact, there is an extensive literature that describes bonds of all maturities by state space models with a small number of factors. The models often impose coefficient restrictions that preclude arbitrage opportunities across different bonds. An arbitrage-free model has the property that any bond price at some date  $t$  can be written as a function of the factor realizations at  $t$ . In particular, affine term structure models imply that all zero coupon bond prices are exponential-affine functions in the factors. We now estimate an affine term structure model with two factors. We then use the model to construct, for every zero coupon bond, a portfolio of three bonds – a short bond and two basic bonds – that replicates closely the return on the given zero coupon bond.

### *Data*

We use quarterly data, so that we interpret one period to be one quarter. The nominal short rate  $i_t$  is therefore the 1-quarter rate. In addition to data on the 1-quarter rate from FRED, we use data on nominal zero-coupon bond yields with longer maturities. These maturities are spaced annually, ranging from 1 year to 20 year maturity. These data come from the McCulloch files available from the website <http://www.econ.ohio-state.edu/jhm/ts/mckwon/mccull.htm>. The sample for these data is 1952:2 - 1990:4. We augment these data with the new Gurkaynak, Sack, and Wright (2006) zero-coupon yields. The resulting sample is 1952:2 - 2004:3.

### *Term structure model*

To get an idea for how many term-structure factors we will need for our quarterly dataset, we compute principal components from our 21 nominal bond yields  $(i_t, i_t^{(4)}, i_t^{(8)}, \dots, i_t^{(80)})$ . The first principal component alone explains 97.3% of the variance in these yields. The second principal component explains another 2.5% of the variance, so that the first two principal components combined capture 99.8% of the variance. The first two principal components have almost one to one correspondences with the short rate and the spread between the longest (20-year) yield and the short rate. The first (second) principal component has a 92.6% (97.5%) correlation with the short

rate (spread). As a consequence, even if the true term structure factors are unobserved, the short rate and the long rate will be good proxies.

We use a term-structure model with factors stacked in the vector  $f_t$ . We use the short rate  $i_t$  and the long rate  $i_t^{(80)}$  as factors and de-mean these two variables before including them in  $f_t$ . We use upper bars to denote sample means and get

$$f_t = \left[ i_t - \bar{i}, i_t^{(80)} - \bar{i}^{(80)} \right]^\top$$

We assume that the factors follow a Gaussian vector autoregression with one lag

$$f_t = \phi_f f_{t-1} + \sigma_f \varepsilon_t$$

with an iid standard normal shock  $\varepsilon_t$  and  $2 \times 2$  matrices  $\phi_f$  and  $\sigma_f$ .

We assume that arbitrage opportunities in bond markets do not exist. This implies the existence of a pricing kernel  $M_t$  which we parametrize in a way that risk premia for bonds will be linear. The pricing kernel is

$$M_{t+1} = \exp \left( -i_t - \frac{1}{2} \lambda_t^\top \lambda_t - \lambda_t^\top \varepsilon_{t+1} \right)$$

where  $\lambda_t$  are the market prices of risk of the two shocks. The vector  $\lambda_t$  is a linear function of the factors

$$\lambda_t = l_0 + l_1 f_t$$

for a  $2 \times 1$  vector  $l_0$  and a  $2 \times 2$  matrix  $l_1$ . We denote the parameters of the model  $\theta = (\phi_f, \sigma_f, l_0, l_1)$ .

We can solve for the price  $P_t^{(n)}$  of a zero-coupon bond with maturity  $n$  by recursively solving the relation

$$P_t^{(n)} = E_t \left[ M_{t+1} P_{t+1}^{(n-1)} \right]$$

with the terminal condition  $p_t^{(0)} = 0$ . The resulting bond prices are exponential linear functions of the factors

$$P_t^{(n)} = \exp \left( A_n + B_n^\top f_t \right)$$

for a scalar  $A_n$  and a  $2 \times 1$  vector  $B_n$  of coefficients that are functions of time-to-maturity. The recursion implies that the coefficients are computed from the following difference equations

$$\begin{aligned} A_{n+1} &= A_n - B_n^\top \sigma_f l_0 + \frac{1}{2} B_n^\top \sigma_f \sigma_f^\top B_n - \bar{i} \\ B_{n+1} &= (\phi_f - \sigma_f l_1)^\top B_n - e_1 \end{aligned}$$

where  $e_1 = [1 \ 0]^\top$ . The initial conditions are given by  $A_0 = -\bar{i}$  and  $B_0 = -e_1$ . The equation for bond prices shows that bond yields  $i_t^{(n)} = -\ln P_t^{(n)}/n$  are linear functions of the factors.

For the model to be internally consistent, we impose the constraint that

$$\begin{aligned} (11) \quad A_{80} &= -80\bar{i}^{(80)} \\ B_{80} &= -80e_2 \end{aligned}$$

where  $e_2 = [0 \ 1]^\top$

We estimate the term-structure model using the following two stage procedure. The parameters  $\theta$  can be partitioned into the parameters  $\phi_f$  and  $\sigma_f$  which govern the factor dynamics and the risk premia parameters  $l_0$  and  $l_1$ . In the first stage, we estimate the VAR parameters  $\phi_f$  and  $\sigma_f$  using standard SUR. In the second stage, we estimate  $l_0$  and  $l_1$  given the VAR estimates from the first stage. This is done by minimizing the sum of squared fitting errors of the model subject to the constraint (11). We compute standard errors for our parameter estimates with GMM, which takes into account the two-stage estimation.

Panel A in Table 1 reports parameter estimates (and will also report standard errors in future drafts.) The estimated dynamics of the factors are highly persistent; the eigenvalues of the matrix  $\phi$  are 0.84 and 0.97. The two factors are positively correlated through the autocorrelation matrix  $\phi_f$  and the volatility matrix  $\sigma_f$ . This is intuitive, since the factors are both interest rates, which tend to comove in the data.

The parameter estimates of  $l_0$  and  $l_1$  govern the behavior of the conditional Sharpe ratio  $-\lambda_t$  (mean excess return divided by standard deviation) on long bonds. Since the standard deviation of excess returns in this model is constant, and the factors  $f_t$  are mean zero, the negative estimate

of  $l_0$  indicates that expected excess returns on long bonds are positive. This can also be seen in Figure 1 which plots the expected returns on the 8 quarter and the 20 quarter bond in excess of the 1-quarter interest rate. The second column of  $l_1$  is negative and indicates that expected excess returns on long bonds are high in periods with high spreads, or in other words in recessions. The model thus captures that expected excess returns are countercyclical, as documented by the regression results of Fama and Bliss (1987). In Figure 1, the pattern emerges clearly. For example, expected excess returns were high in 1991 and 2001.

Panel B reports by how much the model-implied yields differ from observed yields on average. By construction, the model hits the 1-quarter and 80-quarter interest rates exactly, because these rates are included as factors. For intermediate maturities, the error lies within the .20 – .59 percent range. We will see below that these errors are sufficiently small for our purposes.

TABLE 1: ESTIMATION OF TERM STRUCTURE MODEL

Panel A: Parameter Estimates						
$\phi_f$		$\sigma_f \times 100$		$l_0$	$l_1$	
0.884	0.071	0.247	0	-0.115	96.3	-111.6
0.053	0.929	0.065	0.097	-0.050	3.2	-12.1

Panel B: Fitting errors for bond yields (annualized)						
maturity (in quarters)						
	1 qtr	4 qrts	20 qrts	40 qrts	60 qrts	80 qrts
mean absolute errors (in %)	0	0.47	0.59	0.40	0.20	0

*Portfolios to replicate zero coupon bonds*

According to the term structure model, the price  $P_t^{(n)}$  of a zero coupon bond of maturity  $n$  at date  $t$  is well described by  $\exp(A_n + B_n^\top f_t)$ . We now select  $n$  basic bonds, zero coupon bonds with maturity greater than one, and stack their coefficients in a vector  $\hat{A}$  and a matrix  $\hat{B}$ . Our goal is to construct a portfolio containing the basic bonds and the short bond such that the return on the portfolio replicates closely the return on a given zero coupon bond with maturity  $n$ . We use a

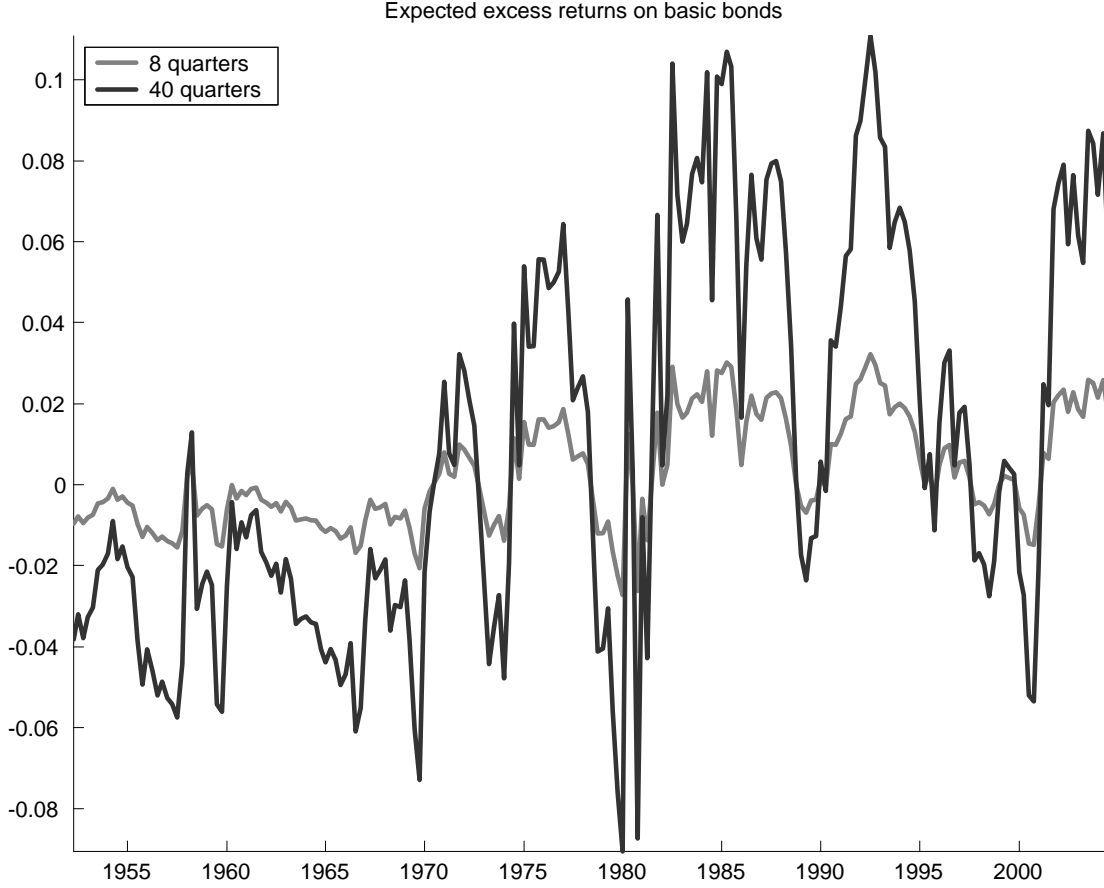


Figure 1: Time variation in expected excess returns

discretization of continuous time returns, similar to those used above for the return on wealth. We approximate the change in price on an  $n$ -period bond by

$$\begin{aligned}
 P_{t+1}^{(n-1)} - P_t^{(n)} &\approx P_t^{(n)} \left( A_{n-1} - A_n + B_{n-1}^\top (f_{t+1} - f_t) + (B_{n-1} - B_n)^\top f_t + \frac{1}{2} B_{n-1}^\top \sigma_f \sigma_f^\top B_{n-1} \right) \\
 &= P_t^{(n)} \left( A_{n-1} - A_n + B_{n-1}^\top (\phi - I) f_t + (B_{n-1} - B_n)^\top f_t + \frac{1}{2} B_{n-1}^\top \sigma_f \sigma_f^\top B_{n-1} \right) \\
 &\quad + P_t^{(n)} B_{n-1}^\top \sigma_f \varepsilon_{t+1} \\
 (12) \quad &= : a_t^{(n)} + b_t^{(n)} \sigma_f \varepsilon_{t+1}
 \end{aligned}$$

Conditional on date  $t$ , we thus view the change in value of the bond as an affine function in the shocks to the factors  $\sigma_f \varepsilon_{t+1}$ . Its distribution is described by  $N + 1$  coefficients: the constant  $a^{(n)}$  and the loadings  $b^{(n)}$  on the  $N$  shocks. In particular, we can calculate coefficients  $(a^{(1)}, b^{(1)})$  for

the short bond, and we can arrange coefficients for the  $N$  basic bonds in a vector  $\hat{a}$  and a matrix  $\hat{b}$ . Now consider a portfolio that contains  $\theta_1$  units of the short bond and  $\hat{\theta}_i$  units of the  $i$ th basic bond. The change in value of this portfolio is also an affine function in the factor shocks and we can set it equal to the change in value of any  $n$ -period bond:

$$(13) \quad \begin{pmatrix} \theta_1 & \hat{\theta}' \end{pmatrix} \begin{pmatrix} a_t^{(1)} & b_t^{(1)} \\ \hat{a}_t & \hat{b}_t \end{pmatrix} \begin{pmatrix} 1 \\ \sigma_f \varepsilon_{t+1} \end{pmatrix} = \begin{pmatrix} a_t^{(n)} & b_t^{(n)} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma_f \varepsilon_{t+1} \end{pmatrix}.$$

Since the  $(N + 1) \times (N + 1)$  matrix of coefficient on the left hand side is invertible for a nondegenerate term structure model, we can select the portfolio  $\begin{pmatrix} \theta_1 & \hat{\theta}' \end{pmatrix}$  to make the conditional distribution of the value change in the portfolio equal to that of the bond.

#### *Replicating portfolios based on a two-factor model*

When stated in terms of units of bonds  $\begin{pmatrix} \theta_1, \hat{\theta} \end{pmatrix}$ , the replicating portfolio for the  $n$ -period bond answers the question: how many short and basic bonds are equivalent to one  $n$ -period bond? For our work below it is more convenient to define portfolio weights that answer the question: how many dollars worth of short and basic bonds are equivalent to one dollar worth of invested in the  $n$ -period bonds? The answer to this question can be computed using the units  $\begin{pmatrix} \theta_1, \hat{\theta} \end{pmatrix}$  and relevant bond prices. Figure 2 provides the answer for the two factor term structure model estimated above. Since the term structure model is stationary, the weights do not depend on calendar time.

The maturity of the  $n$ -period bond to be replicated is measured along the horizontal axis. The three lines are the portfolio weights on the short and the two basic bonds; they sum to one for every maturity  $n$ . As basic bonds, we have selected 8- and 20- quarter bonds. For simplicity, we refer to the basic bonds as the middle and the long bond, respectively. The figure shows that the basic bonds are replicated exactly by portfolio weights of one on themselves. More generally, the replicating portfolios for the most average neighboring bonds. For example, most of the bonds with maturities in between the 1-quarter and 8-quarter bond are generating by simply mixing these two bonds, although there is also a small short positions in the longest bonds. Similarly, most of the bonds with maturities in between the middle and long bond are generating by mixing those two bonds. Intuitively, mixing of two bonds will lead to expected returns that are linear in maturity,

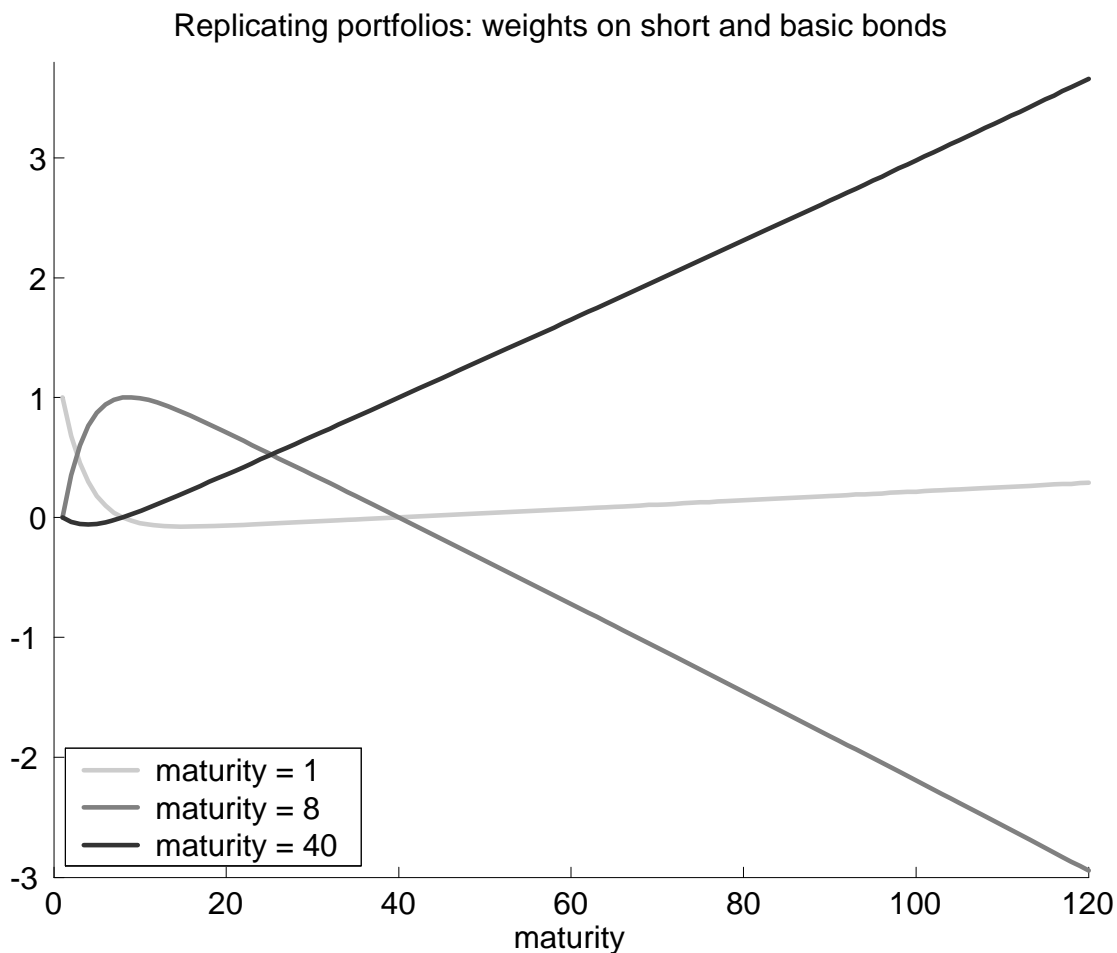


Figure 2: Replicating portfolios; the maturity in quarters of the bond to be replicated is measured along the x-axis.

whereas adding a third bond helps generate curvature.

#### *Quality of the approximation*

We now provide some evidence that the approximation of a zero-coupon bond by a portfolio of short bonds and basic bonds is decent for our purposes. There are two dimensions along which we would like to obtain a good approximation. First, we would like the value of the approximating portfolio to be the same as the value of the zero-coupon bond. This is relevant for measuring the supply of bonds: below we will take existing measures of the quantity of zero-coupon-bonds held by households and convert them into portfolios of the small number of bonds that are tradable by agents in our model. Along this dimension, the approximation is essentially as good as the term

structure model itself. For a replicating portfolio defined by 13, the portfolio value  $e^{-it}\theta_1 + \hat{P}_t^\top \hat{\theta}$  differs from the bond value  $P_t^{(n)}$  only to the extent that the term structure model does not fit bonds of maturity  $n$ . The additional approximation error introduced by the matching procedure is less than .0001 basis points.

Second, we would like the conditional distribution of bond returns to be the same as that of the portfolio return. This is important because we would like agents in our model to have bond investment opportunity sets that are similar to those of actual households who trade bonds of many more maturities. Figure 3 gives an idea about the goodness of the approximation 13 by comparing statistics of the actual return implied by the term structure model and the approximating return. All statistics are unconditional moments computed from our sample, using the realizations of the term structure factors. For example, to obtain the difference in means in the top panel, we compute (i) quarterly returns from the term structure model using the formula  $\exp(A_n + B_n^\top f_t)$  for prices, and (ii) quarterly returns based on the approximate formula for price changes (12) and subtract the mean of (i) from the mean of (ii).

On average, the two return distributions are quite similar for all maturities. The mean returns differ by less than 10 basis points for all bonds shorter than 30 years. The difference in variance is at most 30 basis points. The approximation error increases with maturity, as do the mean return and the variance of returns themselves. The approximate mean return on bonds is always within 5% of the true mean return, while the approximate variance is within 5-15% of the true variance. Larger errors tend to arise for longer bonds. In addition to the univariate distribution of a return, it is also of interest how it covaries with other returns. If the term structure model is correct, then a parsimonious way to check this is to consider the correlation with the two factor innovations. The lower panel of the figure reports the difference between the correlation coefficients of the true and approximate returns with the factor innovations. These differences are very small.

## B. The ROE trading strategy in bonds

To represent the ROE trading strategy in terms of short and basic bonds, we now turn to more complicated fixed-income instruments.

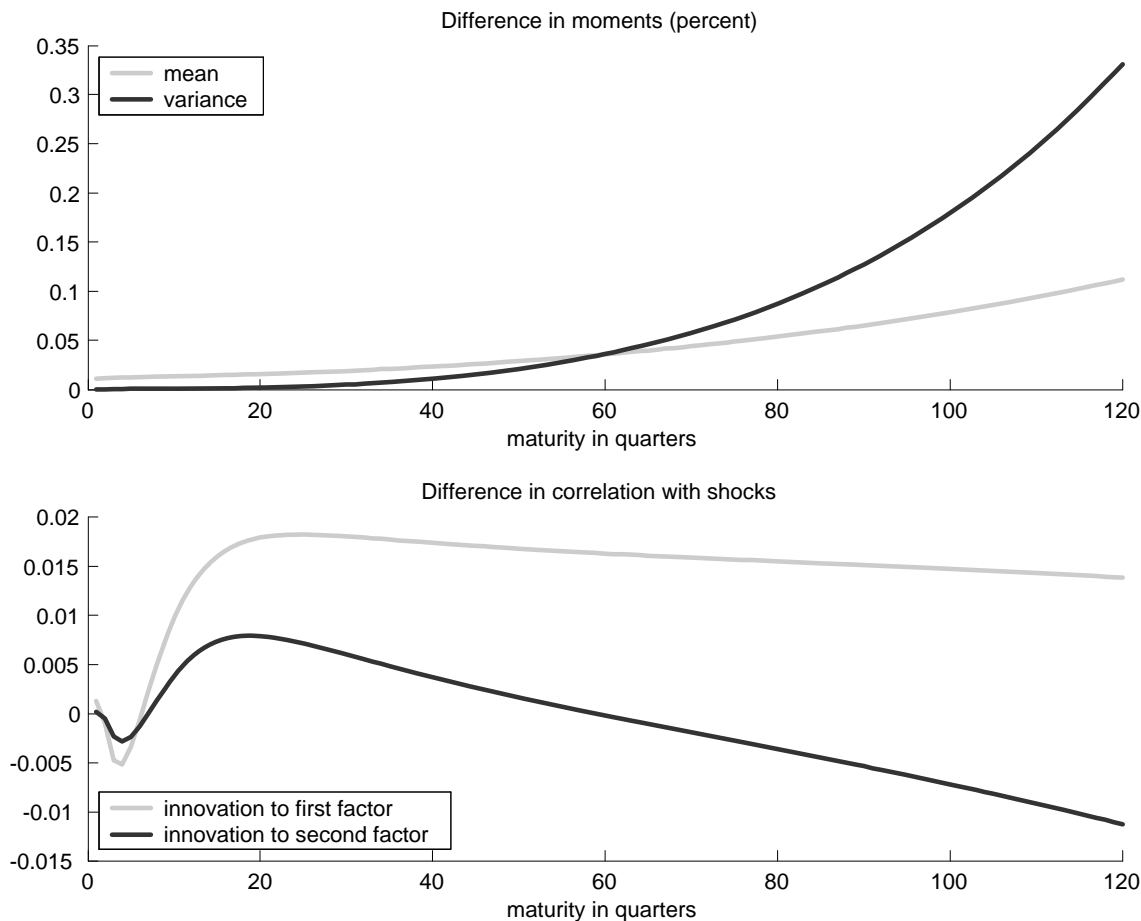


Figure 3: Approximation errors

### *Replicating portfolios for nominal instruments in the Flow-of-Funds*

The Flow-of-Funds (FFA) provides data on book value for many different types of nominal instruments. Doepke and Schneider (2006; DS) sort these instrument into several broad classes, and then use data on interest rates, maturities and contract structure to construct, for every asset class and every date  $t$ , a certain net payment stream that the holders of the asset expect to receive in the future. They use these payment streams to restate FFA positions at market value and assess the effect of changes in inflation expectations on wealth.

Here we determine, for every broad asset class, a replicating portfolio of short and basic bonds. For every asset class and every date  $t$ , DS provide a payment stream, which we can view as a portfolio of zero coupon bonds. By applying (13) to every zero coupon bond, and then summing

up the resulting replicating portfolios across maturities, we obtain a replicating portfolio for the asset class at date  $t$ . Figure 4 illustrates replicating portfolios for Treasury bonds and mortgages. The top panel shows how the weights on the short and basic bonds in the replicating portfolio for Treasury bonds have changed over the postwar period.

The reduction of government debt after the war went along with a shortening of maturities: the weight on the longest bond declined from over 60% in 1952 to less than 20% in 1980. This development has been somewhat reversed since 1980.<sup>7</sup> The bottom panel shows that the effective maturity composition of mortgages was very stable before the 1980s, with a high weight on long bonds. The changes that are apparent since the 1980s are driven by the increased use of adjustable rate mortgages.

We do not show replicating portfolios for Treasury bills, municipal bonds and corporate bonds, since the portfolio weights exhibit few interesting changes over time. All three instruments are represented by essentially constant portfolios of only two bonds: T-bills correspond to about 80% short bonds and 20% middle bonds, that for corporate bonds corresponds to about 60% middle bonds and 40% long bonds, and the replicating portfolio for municipal bonds has 70% long bonds and 30% middle bonds. The final asset class is a mopup group of short instruments, which we replicate by a short bond.

#### *Replicating aggregate FFA household sector positions*

We identify the ROE's issue of bonds at date  $t$  with aggregate household holdings in the FFA at date  $t$ . To derive the ROE trading strategy in short and basic bonds, we thus compute replicating portfolios for household positions in the FFA. One important issue is how to deal with indirect bond positions, such as bonds held in a pension plan or bonds held by a mutual fund, the shares in which are owned by the household sector. Here we make use of the calculations in DS who consolidate investment intermediaries in the FFA to arrive at effective bond positions.

Applying the replicating portfolios for the broad asset classes to FFA household sector positions delivers three time series for the supply of short and basic bonds by the rest of the economy, which are plotted in Figure 5. It is apparent that the early 1980s brought about dramatic changes in US

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<sup>7</sup>The figure shows only the portfolios corresponding to outstanding Treasury *bonds*, not including bills. DS use data from the CRSP Treasury data base to construct a separate series for bills.

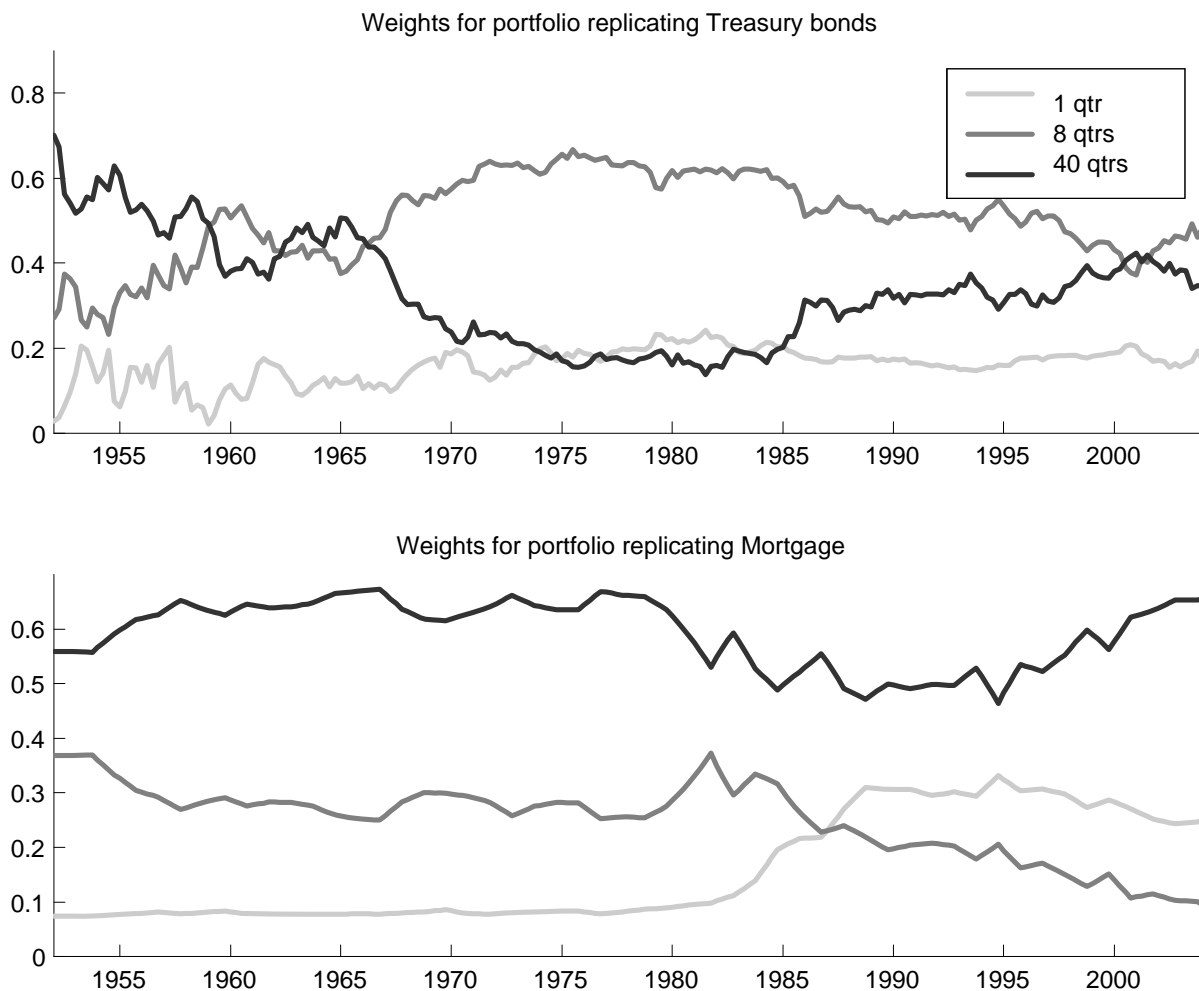


Figure 4: Replicating weights for Treasury bonds and mortgages

bond portfolios. Until then, the supply of short bonds had been trending slightly upwards, whereas the supply of long and middle bonds had been declining. This pattern was reversed during the 1980s and early 1990s. The increase in the share of long bonds was partly due to changes in the composition of Treasury debt, as seen in Figure 4.

### C. Quantifying trading and dividends for the residual asset

We define the residual asset as the sum of claims on businesses, residential real estate, and human capital. We assume that the household sector cannot trade human capital with the rest of the economy. However, trade can take place in equity or housing. We identify the supply of new

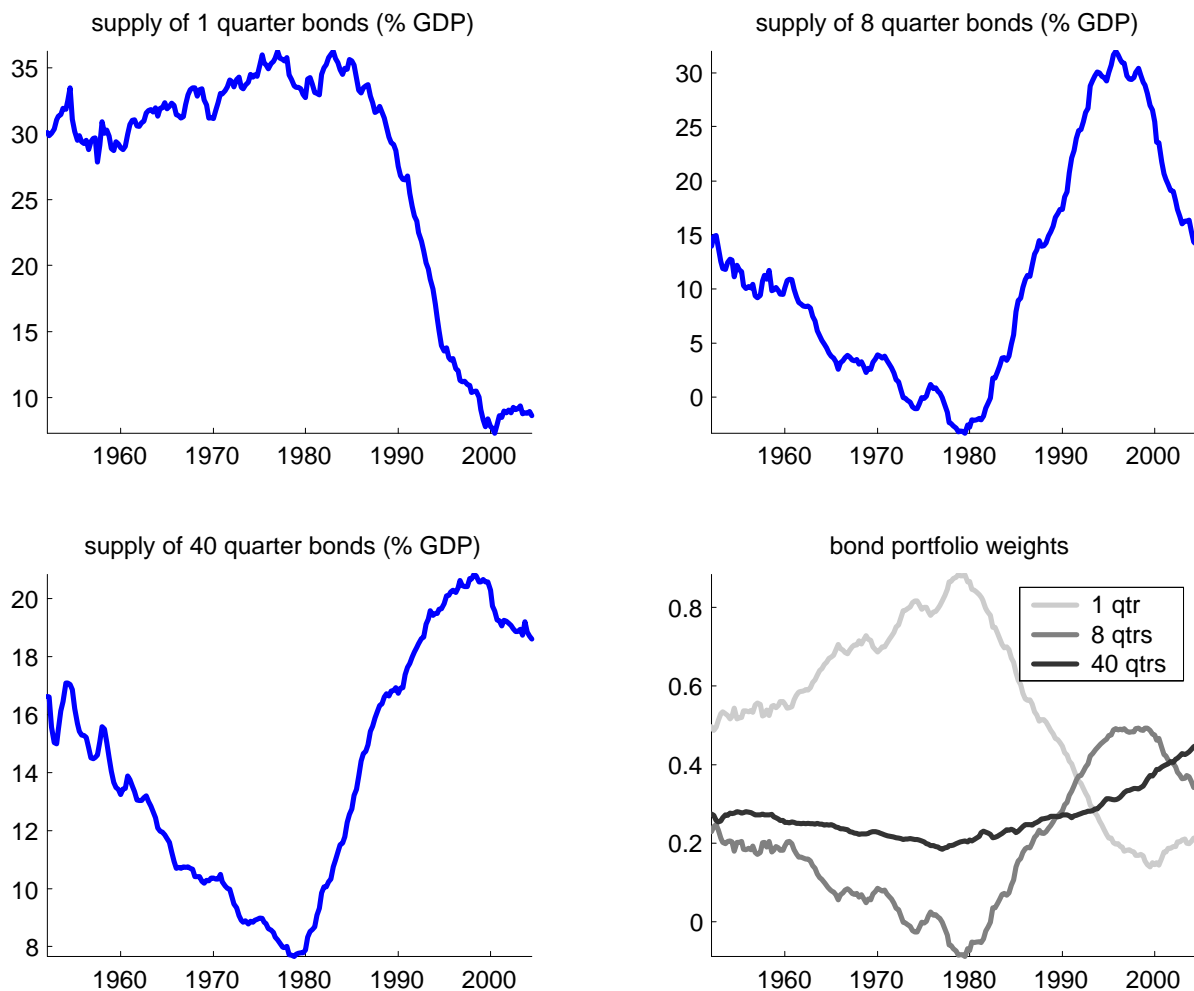


Figure 5: Trading strategy of the ROE in the bond market

residual asset  $Pf_t$  with the sum of net new equity purchases by the household sector plus gross residential investment. Here the new equity purchases (or sales) include equity acquired indirectly through the purchase of mutual fund shares or life insurance policies. The resulting series for the supply of the residual asset is shown in the bottom panel of Figure 6. There is a declining trend, which reflects a reduction in residential investment, as well as increased repurchases of equity and sales of equity to foreigners since 1980.

The dividend on the residual asset is the sum of “dividends” on the component assets in NIPA, shown in the top panel of Figure 6. We include personal dividends on corporate equity, net dividends from noncorporate businesses other than housing, and housing dividends, defined as housing consumption net of maintenance and property tax. We include components of labor income in

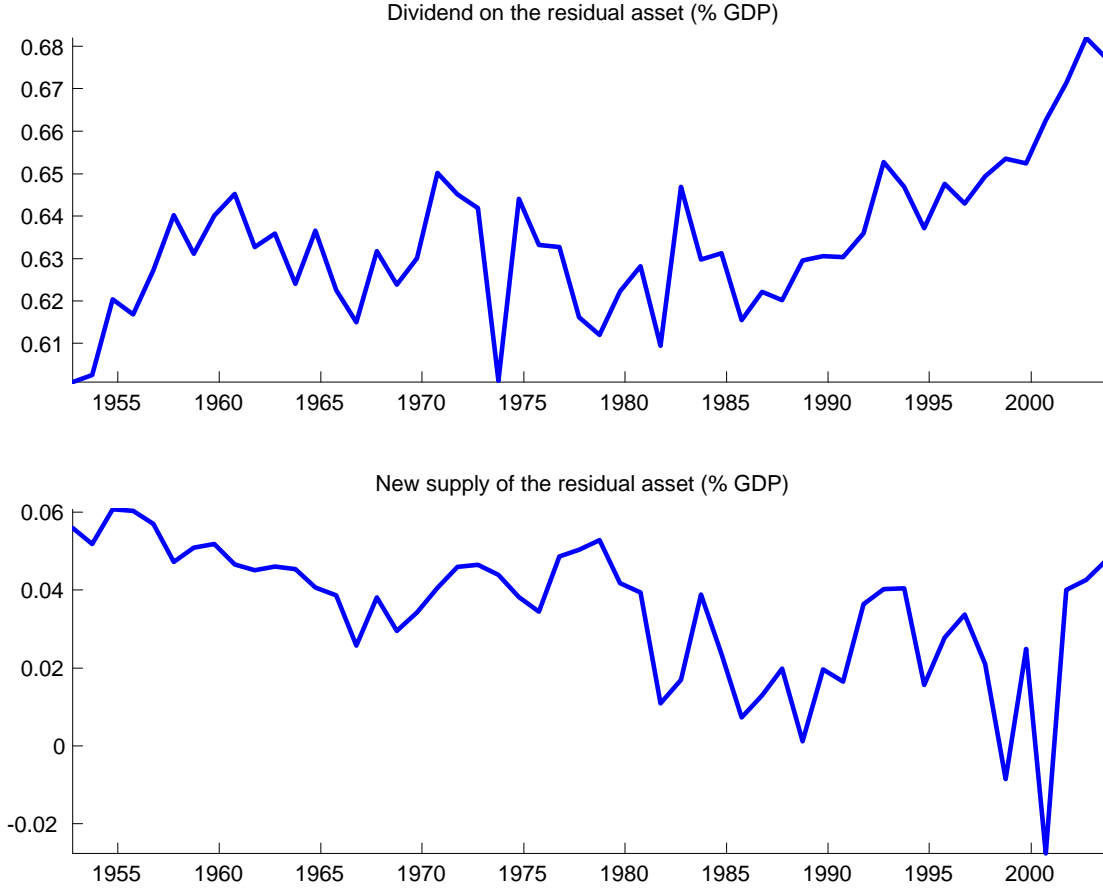


Figure 6: Dividend and ROE trades for the residual asset

NIPA to the extent that they are available for consumption or investment (and hence represent a dividend that should show up in the budget constraint). Of the various components of worker compensation, we include only wages and salaries, as well as employer contributions to DC pension plans. We do not include employer contributions to DB pension plans or health insurance, since these funds are not available for consumption or investment. However, we do include benefits disbursed from DB plans and health plans. Also included are transfers from the government. Finally, we subtract personal income tax on non-asset income.

#### D. VAR for Expectations

To describe the expectations of households, we collect the term structure factors, inflation, and the dividend growth of the residual asset in a vector of observables  $h_t = (f_t, \pi_t, \Delta \ln Y_t)^\top$ . The dynamics

of this vector are

$$\begin{aligned} h_t &= \mu + Cz_{t-1} + e_t \\ z_t &= \phi_z z_{t-1} + \phi_z K e_t, \end{aligned}$$

where  $e_t \sim N(0, \Omega)$ . The vector  $z_t$  is 4-dimensional so that  $z_t = (f_t, z_t^2)^\top$  not only includes the term structure factors  $f_t$  but also additional state variables  $z_t^2$  that are important in forecasting inflation and the return on the residual asset.

For these dynamics to be consistent with our term structure model, we impose the following constraints:

$$\begin{aligned} \mu &= \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} & C &= \begin{pmatrix} \phi_f & 0 \\ 0 & I \end{pmatrix} & \Omega &= \begin{pmatrix} \sigma_f & 0 \\ \sigma_{f2} & \sigma_{22} \end{pmatrix} \\ \phi_z &= \begin{pmatrix} \phi_f & 0 \\ \phi_{f2} & \phi_{22} \end{pmatrix} & K &= \begin{pmatrix} I & 0 \\ K_{f2} & K_{22} \end{pmatrix}, \end{aligned}$$

where  $\mu_2$  is a  $2 \times 1$  vector,  $I$  is a  $2 \times 2$  identity matrix,  $\phi_{f2}$ ,  $\phi_{22}$ ,  $K_{f2}$  and  $K_{22}$  are  $2 \times 2$  matrices.

This parametrization insures that the vector  $f_t$  follows a VAR with one lag, as specified by the term structure model, with autocorrelation matrix  $\phi_f$  and volatility matrix  $\sigma_f$ . It also insures that the third and fourth components of  $z_t$  can indeed be interpreted as expected inflation and expected returns on the residual asset. Since the matrix  $K_{22}$  is not necessarily equal to the identity matrix, these dynamics nest an ARMA(1,1) for inflation. We estimate these dynamics by maximum likelihood, and keep the parameters in  $\phi_f$  and  $\sigma_f$  at their estimated values reported in Table 1.

## IV Equilibrium Interest Rates

In this section, we report numerical results for equilibrium prices.

### *Returns*

Households' return expectations are based on nominal bond payoffs and inflation generated by the VAR estimated in Section III. There is one aspect of beliefs that remains to be specified: beliefs about the return on the residual asset. As a benchmark, we assume that households perceive the return on the residual asset to be i.i.d. and uncorrelated with bond returns. We pin down its mean and variance by calculating, given a discount factor  $\beta$ , the series for equilibrium prices (6). We can use this price series and our dividend series to compute the implied sample mean and variance of the return. Expectations about the residual asset returns are thus consistent with these moments as implied by the model.

The short bond has mean real return 1.48% and one-quarter-ahead conditional volatility .84% in annualized percentage terms. Table 2 summarizes relevant moments of the distribution of excess returns. In terms of conditional Sharpe ratios (mean excess return divided by standard deviation), the residual asset looks most attractive, followed by the middle and the long bond. It is also notable that the long and middle bond returns are highly positively correlated. Neither the basic bonds nor the residual asset covary much with the short bond return. Since the return on the residual asset is i.i.d., any motives for market timing come from the predictability of bond returns, illustrated in Figure 1 above.

### *Preference parameters*

To select preference parameters  $\beta$  and  $\gamma$ , we have searched over a grid to find the best fit for *average* equilibrium yields on the long and short bond over the sample. This search delivers  $\beta = .0997$  and  $\gamma = 3.5$ . The fit for average yields is reported in Table 3. While the model can match the mean nominal short rate, it does not generate spreads that are as large as in the data. The volatilities of the model-implied yields are somewhat lower than those in the data, but they have the right order of magnitude and volatility is decreasing in maturity.

It is worth pointing out how the structure of the model constrains the choice of parameters.

The parameter  $\beta$  fixes the savings rate, and therefore provides the order of magnitude for the price dividend ratio. To see this, assume for the moment that bonds are in zero net supply, and that the rest of the economy does not trade in the residual asset ( $Pf_t = 0$ ). The price dividend ratio is then  $P_t^{res} = \beta / (1 - \beta)$ .<sup>8</sup> More generally, trades with the ROE introduce volatility in  $P_t^{res}$ . However, the scale of the price is still driven by  $\beta$ . As a result, the mean return on the residual asset – which we force household expectations to match – is also driven by  $\beta$ . In particular, it is important that  $\beta$  is large enough, so that the expected return on the residual asset remains small enough for the problem to have a solution.

TABLE 2: MOMENTS OF EXCESS RETURNS (% P.A.)

excess returns (%)	middle bond	long bond	residual asset
mean	.28	.84	3.64
cond. standard deviation	2.1	9.1	10.4
cond. correlation matrix	1	.86	.01
	.	1	-.00
			1
correlation with short bond	-.07	.03	-.08

Given  $\beta$  and hence an expected return for the residual asset, the risk aversion parameter  $\gamma$  can be always be chosen to match the mean short rate. This can be seen from the calculation at the end of Section II: the real interest rate equals the expected return on the residual asset minus the risk premium on that asset, which in turn depends on  $\gamma$ . Quantitatively, our setup forces low values of  $\gamma$  for all feasible values of  $\beta$ . Intuitively, this is because the value of the residual asset is large relative to the value of bonds. This generates a large correlation between the residual asset return and consumption (or the return on wealth) under the optimal plan. Nailing the mean short rate

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<sup>8</sup>This is also the expression of the price-dividend ratio of a consumption claim in the standard Lucas asset pricing model with log utility.

thus means that the risk premium on the residual asset must be kept low.

TABLE 3: MOMENTS OF EQUILIBRIUM YIELDS

nominal yields (% p.a.)		short bond	middle bond	long bond
mean	data	5.27	5.91	6.54
	model	5.27	5.50	6.13
cond. standard deviation	data	1.46	1.45	1.33
	model	1.37	1.24	1.36

Figure 7 plots all equilibrium interest rates together with their data counterparts. The left hand panels show yields, while the right hand panels show spreads, that is, differences between yields. The model tracks quite closely the yield on the two basic bonds. Intuitively, this is because the household has low risk aversion and thus tries to aggressive time the bond market. Shifts in bond demand turn out to be much larger than changes in the trading strategy of the rest of the economy. As a result, there must be large price adjustments in response to changes in bond payoff expectations.

To see the aggressive market timing at work, consider the optimal portfolio rule. Under the approximation 8, portfolio demand (as a percent of total savings) can be written as a linear function in the state vector  $z_t$ :

$$\alpha_t = \begin{pmatrix} 99 \\ 187 \\ -6 \end{pmatrix} + \begin{pmatrix} 1.1 & -0 & 1.1 & .0 \\ -16.2 & 18.1 & -.3 & .1 \\ -.7 & 1.4 & .0 & .0 \end{pmatrix} z_t$$

As in Section II, the three lines correspond to the residual asset, the middle bond and the short bond, respectively. Since the portfolio weights must sum to one, the weight on the short bond is minus the sum of the  $\alpha_t$ s. On average, the agent would like to go short in the lower return short bond and the low-Sharpe-ratio long bond, and invest in the residual asset and the middle bond.

Moreover, investment in the middle responds strongly to the term structure factors, which are

the first two elements of the state vector  $z_t$ . Subsection A. showed that the two factors resemble a short and a long interest rate. It follows that the optimal portfolio strategy invests a lot in the middle bond when the short rate is low and when the long rate is high, or in other words when the yield curve slopes upwards. It is helpful to think about the response to the short rate and the yield spread between long and short rate: holding the short rate fixed, a one percent increase in the spread leads to an 18% higher weight on middle bonds, and, holding the spread fixed, a one percent increase in the short rate leads to a 2% increase in the weight on middle bonds. This strategy is consistent with the conventional wisdom on bond return predictability, and exploits that wisdom very aggressively.

The short rate in Figure 7 is not tracked as closely as the long rates, especially around the high inflation episode of the early 1980s. In fact, the short rate looks “too much” like expected inflation: it is higher than the short rate in the data in the late 1970s, and it is lower in the early 1980s. As a result, movements in spreads over the short rate are also too violent in the middle of the sample. In contrast, the spread between the long and middle bond tracks the data more closely. The behavior of the short rate highlights a potential weakness of two assumptions in the benchmark model: the fact that inflation expectations are derived from a stationary model, and the fact that the return on the residual asset is i.i.d., which – together with low risk aversion – ends up forcing a nearly constant riskless interest rate. One interesting question for further research is to relax these two assumptions.

By plotting unfiltered yield data, Figure 7 shows simultaneously business cycle movements and the trend movements due to the gradual rise and fall of inflation. To distinguish both phenomena, we use a band-pass filter to estimate trend and cyclical components of yields. The filters isolate business-cycle fluctuations in yields that persist for periods between 1.5 and 8 years from those that persist longer than 8 years. Figure 8 plots the trend (thin lines) and cycle (thick lines) for both the data sample (light gray) and the sample of model-implied yields (black lines). The top panel decomposes the short rate, while the middle and lower panels consider the spread between the middle and long bond rate over the short rate, respectively.

The decomposition shows that while the trend components of the short rate differ in the model

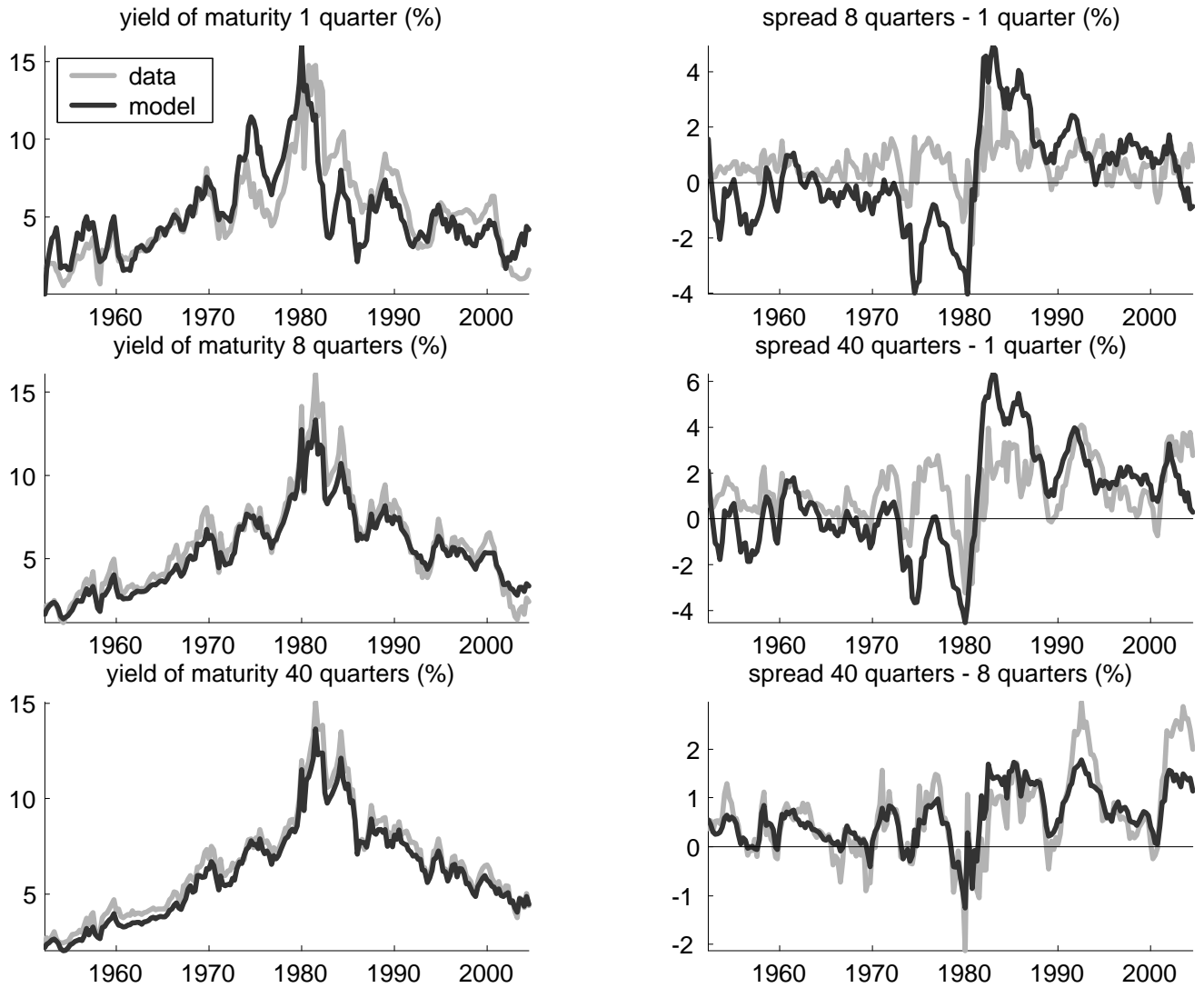


Figure 7: Yields and spreads (as % p.a.) for the data sample (light lines) and the model-implied sample (dark lines).

and the data, business cycle fluctuations in the two series are quite similar. For the 2-year-3-months-spread, the model introduces a low frequency component that is not present in the data, and it overstates business cycle movements in the spread around 1980. For the spread between the middle and long bond, the trend component has the same shape in the model and the data. The model also tracks the business cycle component, although it misses some of the recent business-cycle volatility in spreads.

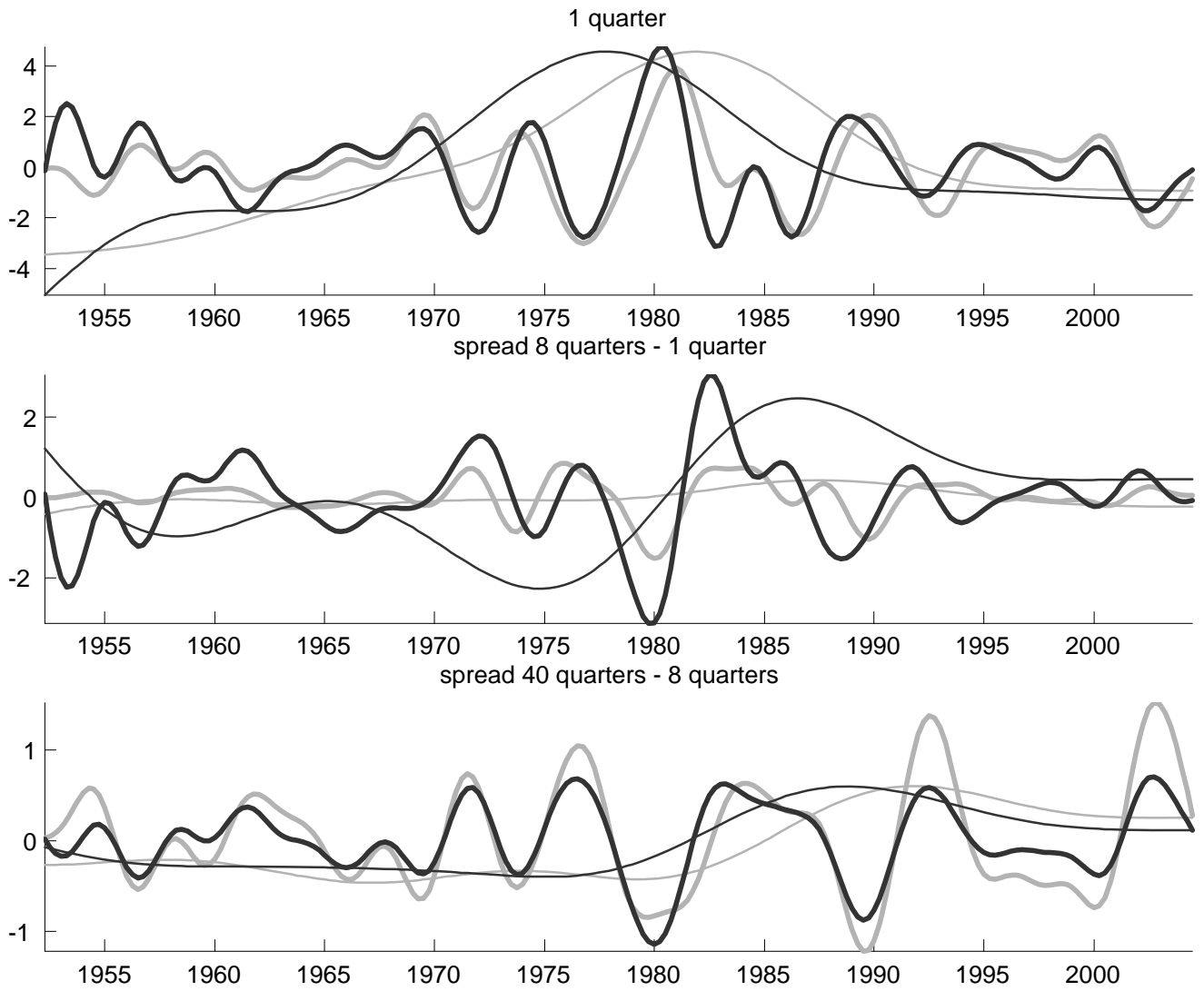


Figure 8: Trend components (thin lines) and cycle component (thick lines) for both the data sample (light gray) and the sample of model-implied yields (black lines). Yields and spreads stated as percent p.a.

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