UNDERSTANDING THE NEW-KEYNESIAN MODEL WHEN MONETARY POLICY SWITCHES REGIMES

ROGER E.A. FARMER, DANIEL F. WAGGONER, AND TAO ZHA

Abstract. This paper studies a New-Keynesian model in which monetary policy may switch between regimes. The study is of substantive importance because the extent to which there is indeterminacy or determinacy in this kind of model is largely unknown. We derive a set of sufficient conditions that enable one to construct a wide range of indeterminate solutions. We show that the necessary and sufficient condition for determinacy, provided by Davig and Leeper (2007), is necessary, but not sufficient. A number of numerical examples are used to illustrate our general point that indeterminacy is much more prevalent than previously thought not only in theory but in practice.

I. Introduction

The *basic new-Keynesian model* (NK) consists of a forward-looking IS curve, an expectations-augmented Phillips curve and a policy rule, in which the interest rate responds to current values of inflation and output. This model is at the core of a wide class of dynamic stochastic general equilibrium (DSGE) models currently in use for policy analysis in both academia and central banks.[1] A monetary policy rule that directs the policy-maker to respond to inflation by raising the interest rate less than one-for-one in response to an increase in inflation is said to be *passive* and a rule that directs the central bank to raise the interest rate more than one-for-one is said to be *active* (Leeper, 1991). A central bank that adopts an active rule is said to follow the *Taylor principle* after work by John Taylor (1993) who argued that a simple rule of this kind is a good characterization of actual central bank policy. In the basic NK

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[1]King (2000) and Woodford (2003) provide good introductions to the basic three-equation new-Keynesian model.

model, passive policies lead to the existence of indeterminate equilibria in the sense that arbitrarily close to one equilibrium there is another one.

It is widely believed that the presence of indeterminacy is undesirable not only because it permits the existence of non-fundamental shocks but also because it amplifies the persistence and volatility of the equilibrium paths of inflation, interest rates, and output in response to *fundamental shocks*.[2] Clarida, Galí, and Gertler (2000), Lubik and Schorfheide (2004), and Boivin and Giannoni (2006) estimate central bank policy rules for the U.S. economy for the period from 1960 through 1996. Their estimates show that macroeconomic volatility has been much lower in the post-1982 period than in the pre-1980 period and they attribute this reduction in volatility to the switch from a passive monetary policy, that implies indeterminacy, to an active policy that implements the unique equilibrium.

Motivated by this empirical work, Davig and Leeper (2007, DL) extend the basic NK model by removing the assumption that a policy rule must be fixed forever. They study two policy regimes, one active in which policy is chosen by an *inflation hawk* and the other passive in which it is chosen by an *inflation dove*. DL allow the coefficients of the Taylor rule to vary stochastically across regimes according to a Markov-switching process and, within this *Markov-switching NK model* (MSNK), they assume that the inflation dove chooses a policy that would lead to indeterminacy if the economy were to remain forever in the passive regime and the inflation hawk chooses a policy that would lead to determinacy under a permanently active regime.[3] DL provide a *necessary and sufficient* condition for determinacy of the equilibrium of the MSNK model and they show that, for this model, the parameter region of determinate equilibria may be considerably larger than the union of the determinacy regions of the two separate NK models where agents do not take account of the probability of future regime change.

The task of finding a necessary and sufficient condition for local indeterminacy (and therefor determinacy), undertaken by Davig and Leeper (2007), is important but challenging. In this paper we re-examine the equilibrium characteristics of the MSNK model and show that whether there is local determinacy in the MSNK model

[2]For a more detailed exposition of this argument see Woodford (2003, page 88).

[3]Economic arguments for modeling policy changes in a probabilistic manner were first put forth by Sims (1982) and by Cooley, LeRoy, and Raymon (1984). These authors argued that once a policy regime has changed, the rational public will expect such shifts to occur again in the future and will form a probability distribution over possible regime change. More recently, Leeper and Zha (2003) have drawn out implications of this way of thinking for practical monetary policy.

when one of the regimes is passive is still an open question. We achieve this conclusion by making four distinct contributions to this literature. First, we derive sufficient conditions for indeterminacy of the MSNK model. Second, we show how to use some of the conditions to construct a wide class of indeterminate solutions. Third, we prove that Davig and Leeper's condition, when used for identifying indeterminacy, is a special case of our general sufficient condition. Moreover, we show that their necessary and sufficient condition turns out to be not sufficient to ensure local determinacy. Fourth, we provide a number of numerical examples to illustrate the practical relevance of our general point: indeterminacy in the MSNK model is much more prevalent than previously thought.

To make our work comparable with the existing literature, we maintain exactly the same assumptions as Davig and Leeper (2007) throughout our paper. Specifically, we assume Ricardian fiscal policy and consider only bounded equilibria. Even with these qualifications, the MSNK model is a much more difficult problem than the basic NK model in important ways since its determinacy and indeterminacy properties depend not only on parameters that describe how policy makers act in any given regime, but also on agents' expectations of the policy parameters in future monetary policy regimes, where regime change is part of the fundamentals.

The rest of our paper is organized as follows. In Section II we present the DL new-Keynesian model with regime switching. In Section III we derive sufficient conditions for indeterminacy that apply to a broader class of Markov switching DSGE models of which the NK model is a member. In Section IV we derive two important corollaries. One corollary shows that the DL condition is a special case of our general sufficient condition for indeterminacy. The other corollary is used to show that the DL condition does not guarantee determinacy of the equilibrium. A number of numerical examples are discussed in Section V to show the practical importance of our theoretical findings. Section VI summarizes our results and makes some suggestions for extensions.

II. The Model

We consider the new-Keynesian DSGE model estimated by Lubik and Schorfheide (2004) and analyzed by Davig and Leeper. This model is described by the following equations,

$$\text{AS curve} \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S, \tag{1}$$

$$\text{IS curve} \quad x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + u_t^D, \tag{2}$$

$$\text{Policy rule} \quad i_t = \alpha_{s_t} \pi_t + \gamma_{s_t} x_t, \tag{3}$$

where x_t is output, π_t is inflation, i_t is the nominal interest rate, u_t^D is an aggregate demand shock, and u_t^S is an aggregate supply shock. Following DL, we measure the variables π_t and i_t as percentage deviations from their steady state values and x_t as the deviation of output from its trend path.

The private sector block, consisting of Equations (1) and (2), has three regime-independent parameters, σ, β and κ. The parameter σ represents the intertemporal elasticity of substitution, β is the discount factor of the representative household, and κ is the slope of the Phillips curve. Uncertain monetary policy is represented by Eq (3), the policy rule. This equation has two regime-dependent parameters (α_{s_t} and γ_{s_t}) that capture the degree to which monetary policy is active or passive. We follow DL and assume that s_t follows an exogenous Markov process with transition matrix $P = [p_{ij}]$. The element p_{ij} represents the probability that $s_t = j$ given $s_{t-1} = i$ for $i, j \in \{1, \dots h\}$ where $h \geq 1$ is the number of regimes. For all theoretical results derived in the paper, h can be any positive integer. For all examples considered in the paper, however, we focus on $h = 2$ where monetary policy is active in the first regime and passive in the second regime.

To write the new-Keynesian model in a compact form, we substitute Eq (3) into Eq (2). Rearranging the terms in Eqs (1)–(2), the model can be written as

$$F_{s_t} y_t = H E_t y_{t+1} + u_t, \tag{4}$$

where

$$y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \qquad u_t = \begin{bmatrix} u_t^S \\ u_t^D \end{bmatrix},$$
$$F_{s_t} = \begin{bmatrix} 1 & -\kappa \\ \sigma^{-1}\alpha_{s_t} & 1 + \sigma^{-1}\gamma_{s_t} \end{bmatrix}, \quad H = \begin{bmatrix} \beta & 0 \\ \sigma^{-1} & 1 \end{bmatrix}.$$

The remainder of the paper is based on this MSNK model and our examples in Section V use the model to make a number of points. The sufficiency theorems and corollaries of Sections III and IV are more general and allow for h regimes and n equations in each regime.

III. Sufficient Conditions for Indeterminacy

In this section we derive sufficient conditions for indeterminacy for a general class of forward-looking Markov-switching rational expectations models that includes the MSNK model as a special case.

Consider models of the form

$$\Gamma_{s_t} y_t = E_t y_{t+1} + \Psi_{s_t} u_t, \tag{5}$$

where y_t is an n-dimensional vector of endogenous random variables and u_t is an m-dimensional vector of exogenous shocks which are allowed to be serially correlated. In particular, u_t may take the vector autoregressive form $u_t = \rho u_{t-1} + \varepsilon_t$ where all the eigenvalues of ρ are strictly less than one in absolute value and ε_t is a bounded exogenous process independent of s_t satisfying $E_{t-1}[\varepsilon_t] = 0$. The new-Keynesian model, Eq (4), is a special case of Eq (5) where $\Gamma_{s_t} = H^{-1} F_{s_t}$ and $\Psi_{s_t} = H^{-1}$.

Davig and Leeper (2007) consider only bounded solutions for y_t. That is, there exists a real number N such that $\|y_t\| < N$ for all t, where $\|\cdot\|$ is any well-defined norm. Throughout this paper, we follow Davig and Leeper and consider only solutions that remain bounded in this sense.

Let $\mathbb{C}^n$ denote the n-dimensional complex space and $diag(X_i)$ denote a block-diagonal matrix whose diagonal elements are $X_1, \ldots, X_h$. We have the following two key theorems. Proofs are collected in Appendix A.

Theorem 1. Let $V_1, \cdots, V_h$ be linear subspaces of $\mathbb{C}^n$ with at least one of the V_i being non-zero and let $m_1, \cdots, m_h$ be positive real numbers. If there exist $n \times n$ complex matrices $\Lambda_{i,j}$ such that

$$\Lambda_{i,j} V_j \subset V_i, \tag{6}$$

$$\|\Lambda_{i,j}\| \leq \frac{m_i}{m_j}, \tag{7}$$

and

$$\Gamma_i v_i = \sum_{j=1}^{h} p_{i,j} \Lambda_{j,i} v_i \text{ for } v_i \in V_i \tag{8}$$

then there exist multiple bounded solutions of Eq (5).

This theorem, although we believe it comes close to giving the full region on which there are multiple solutions, is infeasible to implement in practice. The following theorem, derived from Theorem 1, enables one to construct indeterminate solutions.

Theorem 2. If there exist complex numbers $c_1, \cdots, c_h$ with $|c_i| \leq 1$ for all $i = 1, \ldots, h$ and an nh-dimensional non-zero complex vector v such that

$$\left(diag\left(\Gamma_i\right) - \left(diag\left(c_i\right) P\right) \otimes I_n\right) v = 0, \tag{9}$$

then there exists multiple bounded solutions to Eq (5).

It is computationally feasible to check whether Eq (9) is satisfied by finding c_i's that solve the non-linear equation

$$\det\left(diag\left(\Gamma_i\right)-\left(diag\left(c_i\right)P\right)\otimes I_n\right)=0 \tag{10}$$

subject to the constraint that $|c_i|\leq 1$ for all $i=1,\ldots h$. In the case of one regime this reduces to an eigenvalue problem. In the case of two or more regimes Eq (10) defines a polynomial that can be solved numerically, for example, by a grid search over the h complex numbers $\{c_i\}_{i=1}^{h}$. In the two-regime NK model, if there exists a solution where both of the c_i are inside the unit circle, then Theorem 2 guarantees the existence of multiple bounded solutions to the MSNK model in *both* the active and passive regimes.[4]

To illustrate how to construct an indeterminate solution according to Theorem 2, we concentrate on $h=2$ and assume that u_t follows an i.i.d. process to get a closed-form solution. This i.i.d. assumption does not alter conditions under which the equilibrium is (in)determinate.

The minimum-state-variable solution to the model (5) is given by

$$y_t=G_{s_t}u_t \tag{11}$$

where

$$G_{s_t}=\Gamma_{s_t}^{-1}\Psi_{s_t}.$$

If the sufficiency condition of Theorem 2 is satisfied, the solution is not unique and there are other bounded equilibria in which both inflation and output, represented by y_t, are serially dependent. Lubik and Schorfheide (2004) give a closed-form solution for serially-dependent equilibria for the basic NK model with no regime switching. We show, below, how to find a similar representation of sunspot equilibria for the regime-switching model (5).

Let c_1, c_2, and $v'=[v_1',v_2']$ satisfy Eq (9) of Theorem 2. For this example, the vectors v_1 and v_2 are two-dimensional and since Eq (9) is satisfied, there exist multiple bounded solutions to the regime-switching model, Eq (5), given by the expression,

$$y_t=G_{s_t}u_t+w_t, \tag{12}$$

[4]One can show that for the case $n=1$, Theorem 2 is both necessary and sufficient for the existence of multiple bounded solutions. If one restricts attention to non-negative values for the scalars, Γ_i, then DL's condition for their Fisherian (one dimensional) model becomes a special case of Theorem 2.

where

$$w_t = \left(\frac{c_{s_{t-1}}}{v_{s_{t-1}}^H v_{s_{t-1}}} v_{s_t} v_{s_{t-1}}^H \right) w_{t-1} + v_{s_t} \left(M_{s_t, s_{t-1}, \ldots, s_1} u_t + \xi_t \right), \tag{13}$$

and $v_{s_{t-1}}^H$ denotes the conjugate transpose of $v_{s_{t-1}}$. We have represented the solution as a combination of equilibrium responses to fundamentals $\{s_1, \ldots, s_t, u_1, \ldots, u_t\}$ and non-fundamentals $\{\xi_1, \ldots, \xi_t\}$. For the fundamental component $M_{s_t, s_{t-1}, \ldots, s_1} u_t$, $M_{s_t, s_{t-1}, \ldots, s_1}$ is an arbitrary 1×2 vector that may depend on $s_1, \ldots, s_t$. The non-fundamental component is the sunspot shock ξ_t, which is any one-dimensional bounded stochastic process with zero mean that is independent of both u_t and s_t.[5] The terms c_{s_t} and v_{s_t} are analogous to eigenvalues and eigenvectors in a model with no regime switching. Since the c_{s_t} are all inside the unit circle, the sequence $\{w_t\}$ remains bounded, just as in the one-regime case studied by Lubik and Schorfheide (2003, 2004).

It is straightforward to verify that Eq (12) is a solution. Since the stochastic processes u_t and ξ_t are mean zero and independent of s_t, it can be easily seen that

$$E_t y_{t+1} = \left(\frac{c_{s_t}}{v_{s_t}^H v_{s_t}} E_t v_{s_{t+1}} v_{s_t}^H \right) w_t, \tag{14}$$

where

$$E_t v_{s_{t+1}} = p_{s_t 1} v_1 + p_{s_t 2} v_2. \tag{15}$$

Since $G_{s_t} = \Gamma_{s_t}^{-1} \Psi_{s_t}$, Eq (12) implies

$$\Gamma_{s_t} y_t - \Psi_{s_t} u_t = \Gamma_{s_t} w_t. \tag{16}$$

It follows from Eqs (14), (15), and (16) that Eq (12) is a solution if and only if

$$\left(\frac{c_{s_t}}{v_{s_t}^H v_{s_t}} \left(p_{s_t 1} v_1 + p_{s_t 2} v_2 \right) v_{s_t}^H - \Gamma_{s_t} \right) w_t = 0. \tag{17}$$

Since w_t is proportional to v_{s_t}, Eq (17) needs to be verified only for $w_t = v_{s_t}$. Because c_i and v_i are chosen to satisfy Eq (9), Eq (17) will also hold.

[5]If $v_{s_{t-1}} = 0$, then the coefficient of w_{t-1} is taken to be zero and if any of the c_i are equal to one in absolute value, then M and ξ_t should be taken to be zero in order to guarantee boundedness. If any of the c_i or v_i are complex, then the solution given by Eq (12) will be complex, but either the real or the imaginary component of w_t can be used to construct real solutions.

IV. Two Special Cases

In this section we derive two useful corollaries to Theorem 2. These corollaries concern two special cases of substantive importance. The first corollary shows that Davig and Leeper's (2007) necessary and sufficient condition for determinacy turns out to be only sufficient for indeterminacy (in other words, only necessary for determinacy). The second corollary provides a separate sufficient condition for indeterminacy that can be easily checked in practice. This corollary is important because it can used to construct easily-understood counterexamples to the DL necessary and sufficient condition. Proofs of these corollaries are provided in Appendix B.

Corollary 1. If $diag\left(\Gamma_i^{-1}\right)(P \otimes I_n)$ has an eigenvalue greater than or equal to one in absolute value, then there are multiple bounded solutions to Eq (5).

To see that the condition in this corollary is the same as the DL condition when their condition is used to identify indeterminacy, we first describe the DL condition briefly. DL introduce the random variables π_{1t}, π_{2t}, x_{1t}, and x_{2t}, whose dynamics are determined by the following linear system:

$$\underbrace{\begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & -\kappa \\ \sigma^{-1}\alpha_1 & 0 & 1+\sigma^{-1}\gamma_1 & 0 \\ 0 & \sigma^{-1}\alpha_2 & 0 & 1+\sigma^{-1}\gamma_2 \end{bmatrix}}_{B} \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \\ x_{1t} \\ x_{2t} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \beta p_{11} & \beta p_{12} & 0 & 0 \\ \beta p_{21} & \beta p_{22} & 0 & 0 \\ \sigma^{-1} p_{11} & \sigma^{-1} p_{12} & p_{11} & p_{12} \\ \sigma^{-1} p_{21} & \sigma^{-1} p_{22} & p_{21} & p_{22} \end{bmatrix}}_{A} \begin{bmatrix} E_t\pi_{1t+1} \\ E_t\pi_{2t+1} \\ E_t x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} + \begin{bmatrix} u_t^S \\ u_t^S \\ u_t^D \\ u_t^D \end{bmatrix}. \quad (18)$$

The original variables π_t and x_t are determined by letting $\pi_t = \pi_{it}$ and $\pi_t = x_{it}$ when $s_t = i$ for $i = 1, 2$.

Finding a uniqueness condition for the linear rational expectations system (18) is a standard problem (see, for example, Sims (2002); Lubik and Schorfheide (2003)) and accordingly DL show that a necessary and sufficient condition for determinacy of the equilibrium in Eq (18) is that all the generalized eigenvalues of (B, A) lie inside the unit circle. This condition, however, is *not* the same as finding a unique bounded equilibrium to the original model represented by Eq (4). Corollary 1 and the following proposition imply that the condition of DL is necessary for determinacy, but as we shall see, it is not sufficient.

Proposition 1. At least one of the generalized eigenvalues of (B, A) lies on or outside the unit circle if and only if the condition in Corollary 1 is satisfied.

Proof. At least one of the generalized eigenvalues of (B, A) lies on or outside the unit circle if and only if at least one of the eigenvalues of $B^{-1}A$ lies on or outside the unit circle. Let S be the 4×4 matrix such that multiplication on the right by S interchanges the second and third columns, or equivalently, that multiplication on the left by S interchanges the second and third rows. It can be verified that $SBS = diag(F_i)$ and $SAS = P \otimes H$. Since $S = S^{-1}$ and $\Gamma_i = H^{-1}F_i$, it follows that $B^{-1}A = S(SBS)^{-1}SASS = Sdiag(\Gamma_i^{-1})(P \otimes I_n)S$. Since the eigenvalues of $Sdiag(\Gamma_i^{-1})(P \otimes I_n)S$ are the same as the eigenvalue of $diag(\Gamma_i^{-1})(P \otimes I_n)$, the result follows. □

If at least one of the generalized eigenvalues of (B, A) lies on or outside the unit circle, what is the form of an indeterminate solution allowed by the DL condition? Let λ be a generalized eigenvalue of (B, A) that lies on or outside the unit circle, let v be its generalized eigenvector. For analytic simplicity, assume the demand and supply shocks are i.i.d.. Farmer, Waggoner, and Zha (2008) show that the indeterminate solutions to the DL expanded linear system (18) take the following form:

$$y_t = G_{s_t} u_t + v_{s_t} w_t, \tag{19}$$

where

$$w_t = \lambda^{-1} w_{t-1} + M_{s_t, s_{t-1}, \ldots, s_1} u_t + \xi_t. \tag{20}$$

and $G_i = F_i^{-1}$, v_1 is the first and third component of v and v_2 is the second and fourth component of v. The matrix $M_{s_t, s_{t-1}, \ldots, s_1}$ can be any 1×2 matrix and it could depend on the entire history $\{s_t, s_{t-1}, \ldots, s_1\}$. If λ and v are complex, then this solution will be complex, but both the real and imaginary components will be real solutions.

It is important to note that in addition to the indeterminate solutions represented by Eq (19), the indeterminate solutions represented by (12) are also solutions to the regime-switching model (5), even when all the generalized eigenvalues of (B, A) lie on or outside the unit circle. In particular, the following corollary shows that the DL condition (i.e., all the generalized eigenvalues of (B, A) lie inside the unit circle) is not sufficient for determinacy. Specific counterexamples are presented in Section V.

Corollary 2. If there exists $i \in \{1, \ldots, h\}$ such that Γ_i has an eigenvalue less than or equal to p_{ii} in absolute value, then there are multiple bounded solutions to (5).

Corollary 2 implies that there are conditions based *only* on the parameters of a single regime that imply indeterminacy in *every* regime. This result, which contradicts the DL necessary and sufficient condition for determinacy, is new. It implies that if the inflation dove is sufficiently passive there may be no action that can be taken by the inflation hawk that will restore determinacy *even in the active regime*. Such an indeterminate solution can be constructed as in (12).[6] In the next section, we give a number of practical examples to show that the DL condition is not sufficient for determinacy of the equilibrium.

V. Practical Examples

In the previous sections we show in theory that indeterminacy is more prevalent than previously thought in the literature. It is important to know whether this conclusion matters in practice. In this section we discuss a number of numerical examples showing practical problems of using Corollary 1 as a necessary and sufficient condition for identifying indeterminacy (and hence also for identifying determinacy), as was proposed in the existing literature (Davig and Leeper, 2007). Consider the situation where there is a passive regime that satisfies Corollary 2 but the parameter values are such that Corollary 1 is violated. If one were to use Corollary 1 as though it were a necessary and sufficient condition for indeterminacy, one would conclude that an inflation hawk can make the policy active enough to ensure uniqueness of the equilibrium. But the fact is that there are *no* possible values for the parameters of the interest rate and output-gap response coefficients of the Taylor rule in the active regime that can restore determinacy.

We begin with the baseline parameterization used by DL. Figure 1 corresponds to the upper-right panel of Figure 2 in Davig and Leeper (2007, page 617). It plots α_1 against α_2 when $\gamma_1 = 0, \gamma_2 = 0, \beta = 0.99, \sigma = 1.0$, $\kappa = 0.17$, $p_{11} = 0.8$, and $p_{22} = 0.95$. The dark-shaded area would be regarded as a "determinacy" region by the DL condition if it were used as a necessary and sufficient condition, but Corollary 2 establishes that there exist multiple bounded equilibria at every point in this dark-shaded area.

As another example, we vary the values of p_{11} and p_{22} and plot the indeterminacy region, as shown in Figure 2, using the following parameterization: $\alpha_1 = 3.6, \alpha_2 = 0.7, \gamma_1 = 0.3, \gamma_2 = 0.1, \beta = 0.99, \sigma = 2.84$, and $\kappa = 0.3$. The light shaded area

[6]Farmer, Waggoner, and Zha (2008) show another set of indeterminate solutions that are very similar to (19) except that v is restricted to be the same in both regimes while λ is allowed to depend on the current regime s_t.

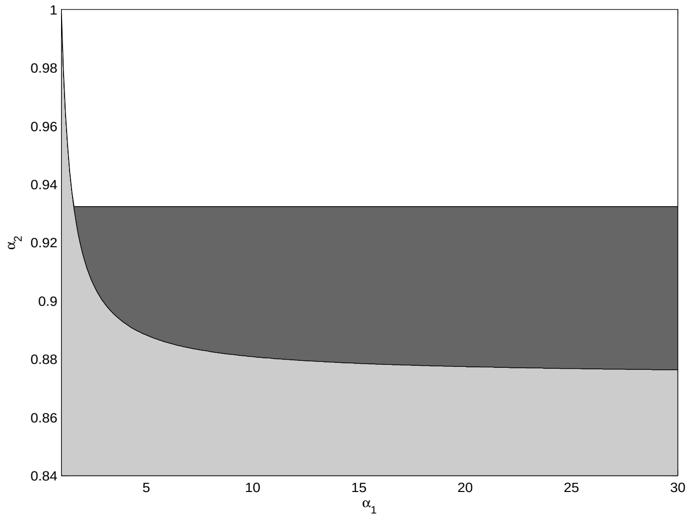

FIGURE 1. Equilibrium characteristics: the new-Keynesian model with $\gamma_1 = 0, \gamma_2 = 0, \beta = 0.99, \sigma = 1.0, \kappa = 0.17, p_{11} = 0.8$, and $p_{22} = 0.95$ (an example of DL's Figure 2). The light shaded area is an indeterminacy region marked by Corollary 1 and the dark shaded area is an additional indeterminacy region according to Corollary 2. The dark-shaded area is regarded as a "determinacy" region by the DL condition.

marks the indeterminacy region by Corollary 1. If the condition in this corollary were used as a necessary and sufficient condition, it would mean that the equilibrium at any point in the dark shaded area is determinate. That is, for example, if the probability of staying in the passive regime (the second regime) is $p_{22} = 0.9$, the equilibrium is indeterminate for $0 < p_{11} \leq 0.57$ and $0.87 \leq p_{11} \leq 1.0$ but would become determinate if the persistence of the active regime happens to be in between (i.e., $0.58 \leq p_{11} \leq 0.86$). But according to Corollary 2, the dark shaded area marks the indeterminate, not determinate, region.

Corollaries 1 and 2 are straightforward to implement because they involve only computing the eigenvalues. Theorem 2 identifies an indeterminacy region that is larger than that identified by its two corollaries. To see whether this difference matters in practice, Figure 3 replicates the upper-right panel of Figure 2 from Davig and Leeper (2007, page 617). This figure plots α_1 against α_2 when $\gamma_1 = 0, \gamma_2 = 0, \beta = 0.99, \sigma = 1.0$, $\kappa = 0.17$, $p_{11} = 0.95$, and $p_{22} = 0.8$. The light-shaded area is identified by Corollary 1. For the most part, the area identified by Corollary 2 overlaps with the light-shaded area (and thus we do not plot it). The dark-shaded area marks the difference between Theorem 2 and its corollaries, and the difference is nontrivial.

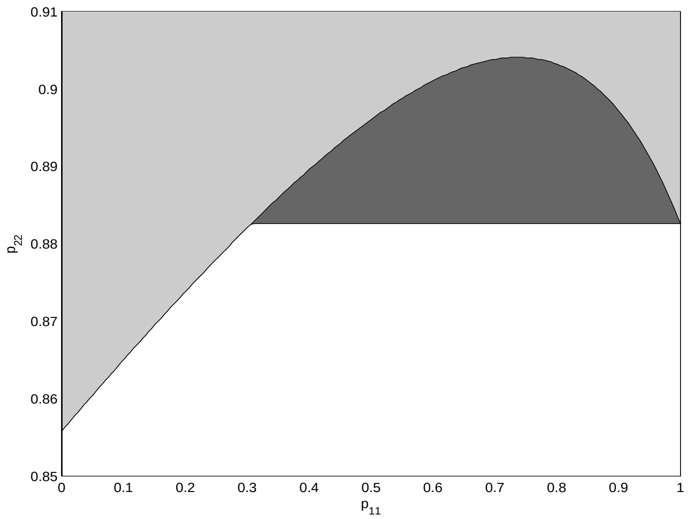

FIGURE 2. Equilibrium characteristics: the new-Keynesian model with $\alpha_1 = 3.6, \alpha_2 = 0.7, \gamma_1 = 0.3, \gamma_2 = 0.1, \beta = 0.99, \sigma = 2.84$, and $\kappa = 0.3$. The light shaded area marks an indeterminacy region by Corollary 1 and the dark shaded area is an additional indeterminacy region according to Corollary 2. The dark-shaded area is regarded as a "determinacy" region by the DL condition.

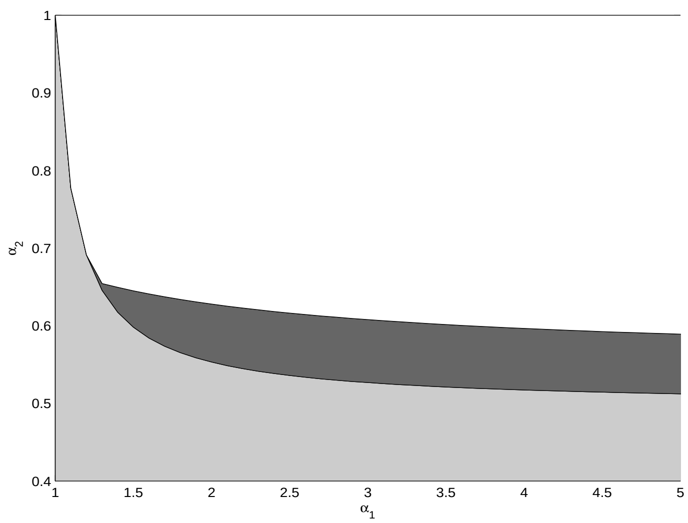

FIGURE 3. Equilibrium characteristics: the new-Keynesian model with $\gamma_1 = 0, \gamma_2 = 0, \beta = 0.99, \sigma = 1.0, \kappa = 0.17, p_{11} = 0.95$, and $p_{22} = 0.8$ (an example of DL's Figure 2). The union of the dark and light shaded areas is an indeterminacy region according to Theorem 2. The dark-shaded area is regarded as a "determinacy" region by the DL condition.

VI. Conclusion

The Taylor rule is widely regarded as an effective way to describe the historical conduct of monetary policy although the parameters of the rule have changed over time as documented by Clarida, Galí, and Gertler (2000), Lubik and Schorfheide (2004), and Boivin and Giannoni (2006). These changes are likely to be embedded in the public's perception that future monetary policy may change for better or worse (Goodfriend, 1993; Sargent, 1999; Mishkin, 2004). What are the equilibrium consequences if the public believes that there is a probability that monetary policy will at times abandon its hawkish stance on inflation in order to accommodate other economic concerns? This question is at the heart of our paper.

The main contributions of the paper have been summarized in the introduction. Here we repeat them briefly. In light of the existing literature (Davig and Leeper, 2007), this paper has made four contributions to the study of the MSNK model. First, we have derived sufficient conditions for local indeterminacy. Second, we have shown how to construct a wide class of indeterminate solutions from these conditions. Third, we have shown that the condition for determinacy, provided in the existing literature, is *insufficient* for guaranteeing determinacy. Finally, we have used a number of numerical examples to illustrate practical implications of our theoretical findings.

A challenging task for future research is to derive a condition for determinacy in the Markov switching new-Keynesian model. The techniques used to construct our examples are easy to implement and will, we hope, point the way to researchers who might wish to take up this challenge.

Appendix A. Proof of Theorems 1 and 2

Because any bounded solution y_{t+1} of (5) can be written as

$$y_{t+1} = \hat{y}_{t+1} + \tilde{y}_{t+1}$$

where $\hat{y}_{t+1}$ is any particular bounded solution of Eq (5) and $\tilde{y}_{t+1}$ is a bounded solution of

$$E_t\left[y_{t+1}\right] = \Gamma_{s_t} y_t, \tag{A1}$$

Eq (5) has multiple bounded solutions if and only if Eq (A1) has a non-zero bounded solution, *assuming a solution of Eq (5) exists.* Since we are interested in the existence of multiple bounded solutions, we will operate under the assumption that there exists at least one bounded solution of Eq (5). This is a rather mild assumption. For instance, Eq (5) will have a solution of the form $y_t = G_{s_t} u_t$ if and only if there is

$nh \times m$ matrix G such that

$$diag\left(\Gamma_i\right) G = \left(P \otimes I_n\right) G\rho + \Psi \tag{A2}$$

where $\Psi' = [\Psi_1' \cdots \Psi_h']$. Eq (A2) is equivalent to the linear equation

$$\left(I_m \otimes diag\left(\Gamma_i\right) - \rho' \otimes P \otimes I_n\right) vec\left(G\right) = vec\left(\Psi\right) \tag{A3}$$

which can be easily solved if $I_m \otimes diag\left(\Gamma_i\right) - \rho' \otimes P \otimes I_n$ is invertible.

Before proceeding with the proof of Theorem 1, we relate the conditions in this proposition to the constant parameter case. Consider the constant parameter analog of Eq (5),

$$E_t\left[y_{t+1}\right] = \Gamma y_t - \Psi u_t. \tag{A4}$$

A bounded solution of Eq (A4) can be characterized by a linear subspace, often referred to as the stable manifold, and a linear reduced form relation that describes the evolution of the solution. The linear subspaces V_i play the role of the stable manifold and the matrices $\Lambda_{i,j}$ play the role of the reduced form coefficients. Equation (6) ensures that the solutions stays on the stable manifold and Equation (8) ensures that we indeed have a solution as long as we are on the stable manifold. Equation (7) guarantees that the solution is bounded. One should note that Eq (7) is stated in terms of a matrix norm, while the more usual conditions for the stable manifold are in terms of eigenvalues. While these conditions are related, they are not the same.

Proof. We inductively construct a non-zero bounded solution of Eq (A1). For $1 \leq i \leq h$, choose $v_i \in V_i$ so that at least one of the v_i is non-zero. Let $y_1 = v_{s_1}$ and $y_{t+1} = \Lambda_{s_{t+1},s_t} y_t$. Condition (6) guarantees that $y_t \in V_{s_t}$. This, together with condition (8), yields

$$\begin{aligned} E_t\left[y_{t+1}\right] &= \sum_{j=1}^{h} p_{s_t,j} E_t\left[y_{t+1} | s_{t+1} = j\right] \\ &= \sum_{j=1}^{h} p_{s_t,j} \Lambda_{j,s_t} y_t \\ &= \Gamma_{s_t} y_t. \end{aligned}$$

So y_t is a solution of Eq (A1). This solution is bounded since, using condition (7),

$$\begin{aligned} \left\| y_{t+1} \right\| &= \left\| \Lambda_{s_{t+1},s_t} \Lambda_{s_t,s_{t-1}} \cdots \Lambda_{s_2,s_1} v_{s_1} \right\| \\ &\leq \left\| \Lambda_{s_{t+1},s_t} \right\| \left\| \Lambda_{s_t,s_{t-1}} \right\| \cdots \left\| \Lambda_{s_2,s_1} \right\| \left\| v_{s_1} \right\| \\ &\leq \frac{m_{s_{t+1}}}{m_{s_1}} \left\| v_{s_1} \right\|. \end{aligned}$$

While we have constructed a non-zero bounded solution, it could be a complex. However, in this case, both the real and imaginary components will be bounded solutions of Eq (A1) and at least one will be non-zero. □

Theorem 2 is essentially Theorem 1 applied to the case in which each of the V_i are one-dimensional. We now proceed with the proof of Theorem 2

Proof. Suppose that there exists v and c_i as in the Theorem 2 and let v_i be the i^{th} n-dimensional block of v. Define V_i to be the subspace spanned by v_i and define $\Lambda_{i,j}$ and m_i by

$$\Lambda_{i,j} = \begin{cases} \frac{c_j}{\|v_j\|^2} v_i v_j^H & v_j \neq 0 \\ 0 & v_j = 0 \end{cases} \text{ and } m_i = \begin{cases} \|v_i\| & v_i \neq 0 \\ 1 & v_i = 0 \end{cases},$$

where v_j^H is the the conjugate transpose of v_j. By construction, $\Lambda_{i,j}$ satisfies condition (6). Also,

$$\|\Lambda_{i,j}\| = \begin{cases} |c_j| \frac{m_i}{m_j} & v_j \neq 0 \\ 0 & v_j = 0 \end{cases}.$$

In either case, $\|\Lambda_{i,j}\| \leq \frac{m_i}{m_j}$ so condition (7) holds. Finally, note that Eq (9) implies

$$\begin{aligned} \Gamma_i v_i &= (e_i' \otimes I_n)\, diag\,(\Gamma_i)\, v \\ &= (e_i' \otimes I_n)\, diag\,(c_j)\,(P \otimes I_n)\, v \\ &= c_i \sum_{j=1}^{h} p_{i,j} v_j \\ &= \sum_{j=1}^{h} p_{i,j} \left(\frac{c_i}{\|v_i\|^2} v_j v_i^H \right) v_i \\ &= \sum_{j=1}^{h} p_{i,j} \Lambda_{j,i} v_i, \end{aligned}$$

where e_i is the i^{th} column of the $h \times h$ identity matrix. So condition (8) holds. □

Appendix B. Proof of Corollaries 1 and 2

Proof. If (λ, u) is a eigenvalue-eigenvector pair of $diag\left(\Gamma_i^{-1}\right)(P \otimes I_n)$ with $|\lambda| \geq 1$, then

$$\left(\lambda I_{nh} - diag\left(\Gamma_i^{-1}\right)(P \otimes I_n)\right) u = 0$$

or

$$\left(diag\,(\Gamma_i) - \frac{1}{\lambda}(P \otimes I_n) \right) u = 0.$$

Corollary 1 follows from Theorem 2 by taking $c_i = 1/\lambda$ and $v = u$. □

Proof. Suppose (λ, u) is a eigenvalue-eigenvector pair of Γ_i with $|\lambda| \leq p_{i,i}$. Define

$$c_j = \begin{cases} \frac{\lambda}{p_{i,i}} & j = i \text{ and } p_{i,i} > 0 \\ 0 & otherwise \end{cases}, \; v_i = \begin{cases} u & j = i \\ 0 & j \neq i \end{cases} \text{ and } v = \begin{bmatrix} v_1 \\ \vdots \\ v_h \end{bmatrix}.$$

An easy calculation shows that

$$(diag\,(\Gamma_i) - (diag\,(c_i) \otimes I_n)\,(P \otimes I_n))\,v = 0,$$

and so the second corollary follows from Theorem 2. $\square$

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